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Criteria of Existence for a *q* Fractional *p*-Laplacian Boundary Value Problem

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This paper is devoted to establishing some criteria for the existence of non-trivial solutions for a class of fractional q-difference equations involving the p-Laplace operator, which is nowadays known as Lyapunov's inequality. The method employed for it is based on a construction of a Green's function and its maximum value. Parallel to this result, it is worth mentioning that the Hartman-Wintner inequality for the q-fractional p-Laplace boundary value problem is also provided. It covers all previous results known in the literature on the fractional case as well as that on the classical ordinary case. The non-existence of non-trivial solutions to the q-difference fractional p-Laplace equation subject to the Riemann-Liouville mixed boundary conditions will obey such integral inequalities. The tools mainly rely on an integral form of the solution construction of a Green function corresponding to the considered problem and its properties as well as its maximum value in consideration where the kernel is the Green's function. The example that we consider here for applying this result is an eigenvalue fractional problem. To be more specific, we provide an interval where an appropriate Mittag-Leffler function to the given eigenvalue fractional boundary problem has no real zeros.

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1. INTRODUCTION

The field of fractional calculus and its applications to the class of partial differential equations, as well as ordinary equations, gained a rapid development. Interesting fractional results turn, in general, on the existence and non-existence of solutions. Such kinds of fractional equations come from different disciplines in sciences, covering medical and engineering matters. Techniques used in this kind of work recourse mainly to the use of Green's function and its corresponding maximum values, which is not always an easy approach. The fractional differential equations with the p-Laplacian operator involves this mathematical tool. However, to overcome this kind of difficulty, another approach can be taken, namely the use of the Cauchy-Schwarz inequality and related inequalities as holders. Different aspects have been considered by many different researchers in this respect. They have treated the existence of either a single or multiple solutions for linearity, but there have been few cases for non-linearity. In addition to this, the manner to extend these results to a general case with more a general operator seems non-evident and requires a thorough analysis of the maximum value of Green's functions. This paper is devoted to tackling this problem with the p-Laplace operator using the Green's function method for the non-linearity case.

Some results focusing on the existence of positive solutions of boundary value problems for a class of fractional differential equations with the p-Laplacian operator have been raised in previous papers (see [1–22] and the references therein). Ren and Chen [15] and Su et al. [17] established the existence of positive solutions to four-point boundary value problems for non-linear fractional differential equations with the p-Laplacian operator. However, for papers on this line concerning the q-difference type of fractional problems, we refer the reader to references [1–4, 8–12, 15, 18, 19, 19–33].

It is worthy of notice that the q-fractional calculus was introduced by Jackson [30, 31], as the reader may observe in consulting the article of Ernst [28], where he attributed the work to Jackson.

Accordingly, we mention the recent developments related to this subject (see [5, 12, 13, 32, 34–46]) and the references therein.

For multiple solutions for the non-linear case, we refer to the work done by El-Shahed and Al-Askar [47], whereas Graef et al. [48] deal with positive solutions by applying different methods.

The first result came from Liapunov [6], in the second ordinary differential equation. It was shown that if u is a non-trivial solution of

$$\begin{cases} u''(t)+q(t)u(t)=0, \quad a<t<b \\ u(a)=u(b)=0, \end{cases}$$

where $a<b$, a and b are two real constants, and the function $q\in C([a,b];\mathbb{R})$, then the function q must satisfy the following integral inequality:

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a}. \tag{1}$$

After this result, several extensions are derived from this one, and consequently, analogous inequalities are obtained for a class of fractional differential equations subject to different kind of boundary conditions (see [5, 12–14, 29, 32, 34–36, 39–41, 44, 45, 49, 50]). However, concerning the fractional q-difference boundary value problem, it was shown in Jleli and Samet [42] that a non-trivial solution of

$$\begin{cases} {}_aD_q^{\alpha}u(t)+Q(t)(t)u(t)=0, \quad t\in(a,b),\ q\in[0,1),\ 1<\alpha\le 2, \\ u(a)=0, \quad u(b)=0, \end{cases} \tag{2}$$

where ${}_aD_q^{\alpha}$ denotes the fractional q-derivative of Riemann-Liouville type [43, 51], and $Q:[a,b]\to\mathbb{R}$ is a continuous function, exists if the following integral inequality

$$\int_a^b (s-a)^{\alpha-1}(b-(qs+(1-q)a)_a{}^{(\alpha-1)}\,|Q(s)|\,{}_ad_qs \ge \Gamma(\alpha)(b-a)^{\alpha-1} \tag{3}$$

is satisfied.

In the opinion of the authors, there are no articles dealing with these types of inequalities for the study of non-trivial solutions for the p-Laplacian operator involving the q-fractional case. We therefore fill the gap in the literature with this paper.

Our result generalizes that one investigated in Jleli and Samet [42].

In this work, we aim to investigate the following q-fractional boundary value problem with the p Laplace operator

$$\begin{cases} {}_aD_q^{\beta}(\phi_p({}_aD_q^{\alpha}u(t))+Q(t)\phi_p(u(t))=0, \quad t\in(a,b), \\ u(a)=0, \quad u(b)=Au(\xi), \\ {}_aD_q^{\alpha}(a)=0, \quad {}_aD_q^{\alpha}u(b)=B\,{}_aD_q^{\alpha}u(\delta), \end{cases} \tag{4}$$

where ${}_aD_q^{\alpha}$, ${}_aD_q^{\beta}$ are the fractional q-derivative of the Riemann-Liouville type with $1<\alpha,\ \beta<2$, $0\le A,\ B\le 1$, $0<\xi,\ \delta<1$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_r$, $\frac{1}{p}+\frac{1}{r}=1$, and $Q:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$.

We prove that the necessary condition of the existence of non-trivial solutions of (4) is the following:

$$1\le\left(\int_a^b \tilde{G}_q(s)\,{}_ad_qs\right)\left(\int_a^b \tilde{H}_q(s)\,{}_ad_qs\right), \tag{5}$$

where $\tilde{G}_q(s)$ and $\tilde{H}_q(s)$ are defined respectively by

$$\tilde{G}_q(s) := \frac{1}{\Gamma_q(\alpha)}\frac{q(s-a)^{(\alpha-1)}}{(b-a)^{\alpha-1}}\left(b-(qs+(1-q)a\right)_a^{(\alpha-1)} + \frac{Ag(\xi,(qs+(1-q)a)(b-a)^{\alpha-1}}{\gamma}, \tag{6}$$

$$\tilde{H}_q(s) := \frac{1}{\Gamma_q(\beta)}\frac{q(s-a)^{(\beta-1)}}{(b-a)^{\beta-1}}\left(b-(qs+(1-q)a\right)_a^{(\beta-1)} + \frac{A\,h(\xi,(qs+(1-q)a)(b-a)^{\beta-1}}{\bar{\gamma}}, \tag{7}$$

where

$$\gamma := (b-a)^{\alpha-1}-A\,(\xi-a)^{\alpha-1},$$

and

$$\bar{\gamma} := (b-a)^{\beta-1}-b(\delta-a)^{\beta-1}.$$

Besides, we show that from this inequality derive several existing previous results in the literature as well as the standard Lyapunov inequality (1): those of Hartman and Wintner [52], Ferreira [39], and so on.

2. DEFINITIONS AND LEMMAS

In this section, we adopt the main tools that will be needed in the subsequent sections; these belong to q fractional calculus. Notations, definitions and lemmas are recalled in order to cover the goal of this paper, whereas, for consistency, we conserve the same notations for q fractional material as adopted in Jleli and Samet [42].

Let $q \in (0,1)$, $N_0 = \{0,1,2,\dots\}$, and define

$$[a]_q = \frac{q^a - 1}{q-1}, \quad a \in \mathbb{R}.$$

The similar q formula to the power $(a-b)^n$ with $n \in N_0$ is

$$(a-b)^0 = 1, \qquad (a-b)^n = \prod_{k=0}^{n-1}(a - bq^k), \quad n \in N,\ a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^\alpha = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

For the particular case when $b = 0$, we note $a^{(\alpha)} = a^\alpha$. Also, the similar q formula to the power function $(x-y)^n$, with $n \in N_0$ is

$$(x-y)_a^{(0)} = 1, \quad (x-y)_a^{(k)} = \prod_{i=0}^{k-1}\left((x-a) - (y-a)q^i\right),\ k \in N,\ (x,y) \in \mathbb{R}^2.$$

For the general case, when $\gamma \in \mathbb{R}$, then

$$\begin{aligned}(x-y)_a^{(\gamma)} &= (x-a)^\gamma \prod_{i=0}^{k-1}\left(\frac{(x-a)-(y-a)q^i}{(x-a)-(y-a)q^{\gamma+i}}\right), \qquad (8)\\ &= (x-a)^\gamma \prod_{i=0}^{k-1}\left(\frac{1 - q^i\frac{(y-a)}{(x-a)}}{1 - q^{\gamma+i}\frac{(y-a)}{(x-a)}}\right)\\ &= (x-a)^\gamma \left(1 - q\frac{(y-a)}{(x-a)}\right).\end{aligned}$$

It has the following properties

- $(t-s)_q^{(\beta+\gamma)} = (t-s)_q^{(\beta)}(t-qs)_q^{(\gamma)}$
- $(at - as)_q^{(\beta)} = a^\beta\,(t-s)_q^{(\beta)}$.

When derivatives are involved, it holds:

- $(t-a)^\alpha \geq (t-b)^\alpha$, for $a \leq b \leq t$, and $\alpha > 0$.

We define the q-Gamma function by

$$\Gamma_q(x) = \frac{(1-q)_0^{(x-1)}}{(1-q)^{x-1}}, \quad x \in R\backslash\{0,-1,-2,-3,\dots\}.$$

In particular one has

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x),\ \forall x > 0,\ \Gamma_q(1) = 1.$$

Here and further, we recall some properties of the q-fractional derivative of a function f defined on $[a,b]$, $a < b$, to $\mathbb{R}$.

The q-fractional derivative of a function $f:[a,b] \to \mathbb{R}$, is defined by

$$({}_aD_qf)(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)},\ t \neq a,$$

and

$$({}_aD_qf)(a) = \lim_{t\to a}({}_aD_qf)(t).$$

Remark:

By using the following changes:

$$q := \frac{x-a}{y-a},$$

it is easy to conclude that if $({}_aD_qf)(t) \leq 0$ (respectively, $({}_aD_qf)(t) \geq 0$) then f is decreasing (respectively, f is increasing).

Remark:

If f is differentiable in (a,b) then

$$\lim_{q\to 1^-}(aD^qf)(t) = f'(t).$$

The q-fractional derivative of a function $f:[a,b] \to \mathbb{R}$ of higher order is defined by

$$({}_aD_q^0f)(t) = f(t), \quad \text{and} \quad ({}_aD_q^nf)(t) = ({}_aD_q(({}_aD_q^{n-1}f)(t)),\ n \in N.$$

The q-derivative of a product and a quotient of functions f and g defined on $[a,b]$ follows as

$$\left({}_aD_q fg\right)(t) = f(t)({}_aD_qg)(t) + g(qt + (1-q)a)({}_aD_qf)(t),$$

and

$$({}_aD_q\frac{f}{g})(t) = \frac{({}_aD_qf)(t)g(t) - ({}_aD_qg)(t)f(t)}{g(t)g(qt + (1-q)a)}.$$

Lemma 2.1. [44] *For $t, s \in [a,b]$, the following formulas hold:*

$${}_t({}_aD_q(t-s)_a^{(\gamma)}) = [\gamma]_q(t-s)_a^{(\gamma-1)},$$

and

$${}_s({}_aD_q(t-s)_a^{\gamma}) = -[\gamma]_q(t - (qs + (1-q)a))_a^{\gamma-1},$$

where ${}_i({}_aD_q)$ denotes the q-derivative with respect to the variable i.

Remark:

If $\gamma > 0$, $a \leq b \leq t$, then

$$(t-a)_0^{(\gamma)} \geq (t-b)_0^{(\gamma)}.$$

Next, we recall the q-integral of a function f defined on $[a,b]$, $a < b$, to $\mathbb{R}$ and its properties.

The q-integral of a function $f:[a,b] \to R$ is defined by

$$\begin{aligned}({}_aI_q^0f)(t) &= \int_a^t f(s)\,{}_ad_qs = (1-q)(t-a)\Sigma_{i=0}^{\infty}q^if(q^it\\ &\quad + (1-q)a),\ t \in [a,b].\end{aligned}$$

One may see that the above series is convergent if f is continuous.

If $a < c < b$, then the following integral equality is satisfied

$$\int_c^t f(s)\,{}_a d_q s + \int_a^t f(s)\,{}_a d_q s = \int_a^c f(s)\,{}_a d_q s,\ t \in [a,b].$$

The following two relations are also satisfied

$$({}_a I_q^0 f)(t) = f(t), \quad \text{and } ({}_a I_q^n f)(t) = {}_a I_q({}_a I_q^{n-1} f)(t),\ n \in N.$$

An essential and important theorem that is known for the classical ordinary case is also valid for the fractional one; it is the fundamental theorem of calculus. Once applied to the fractional operator, we get

$$({}_a D_q\, {}_a I_q f)(t) = f(t) - f(a)$$

if the continuity of the function f is provided. When the continuity of f is avoided, we obtain

$$({}_a D_q\, {}_a I_q f)(t) = f(t).$$

Another crucial integration that is very useful in dealing with non-existence of solutions for a class of fractional boundary value problems is the integration by parts. It follows as

$$\int_a^b f(s)({}_a D_q g)(s)\,{}_a d_q s = [f(t)g(t)]_{t=a}^{t=b} - \int_a^b g(qs + (1-q)a)({}_a D_q f)(s)\,{}_a d_q s.$$

The rule of q-integration by parts is also expressed by (see [24])

$$\int_0^a g(t) D_q f(t)\, d_q t = fg(a) - \lim_{n\to+\infty} fg(aq^n) - \int_0^a D_q g(t) f(qt)\, d_q t. \tag{9}$$

If and g are q-regular at zero, then the limit on the right-hand-side of (9) can be replaced by $(fg)(0)$. (For more details, see [24]).

In what follows, we define the q-fractional Riemann-Liouville integral of a function f defined on $[a,b]$ as follows

$$({}_a I_q^0 f)(t) = f(t).$$

Let us assume that f and g are two functions defined on $[a,b]$ such that $f \leq g$, then the following properties are satisfied

$$\int_a^b f(s)_a d_q s \leq \int_a^b g(s)_a d_q s,$$

and

$$\int_a^b f(s)_a d_q s \leq \int_a^b |f|(s)_a d_q s.$$

As auxiliary results, we need to use the following two lemmas. The reader may consult [23, 30, 31] for more details.

Lemma 2.2. [30, 50] *Let $f:[a,b] \to \mathbb{R}$ be a continuous function. Then*

$$(i)\ {}_a D_q^\alpha ({}_a I_q^\alpha f)(t) = f(t),\ \alpha > 0,\ t \in [a,b],$$
$$(ii)\ {}_a I_q^\alpha\, {}_a I_q^\beta f(t) = {}_a I_q^{\alpha+\beta} f(t),\ \alpha, \beta > 0,\ t \in [a,b].$$

Lemma 2.3. [30, 50] *Let $\alpha > p - 1$ and p be a positive integer. The following then holds:*

$$({}_a I_q^\alpha)\, {}_a D_q^p f(x) = ({}_a D_q^p)\, {}_a I_q^\alpha f(x) - \Sigma_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha + k - p + 1)}\, {}_a D_q^k f(a).$$

2.1. Results and Consequences

The method that we would like to apply here consists of getting an equivalent integral representation of the non-trivial solution of the considered fractional boundary value problem. It therefore necessitates an appropriate construction of the Green function, which plays a crucial role in getting Lyapunov's inequalities.

In order to reduce the q fractional boundary value problem (4) to an equivalent integral equation, an auxiliary result is needed. It is formulated in the following Lemma.

Lemma 2.4. *Let $u \in AC([a,b])$. The unique non-trivial solution of the q fractional boundary value problem*

$$\begin{cases} {}_a D_q^\alpha u(t) + Q(t)z(t) = 0, & t \in (a,b), \\ u(a) = 0, \quad u(b) = Au(\epsilon), \end{cases}$$

where $1 < \alpha < 2$, $a < \epsilon < b$, and $0 \leq A \leq 1$, is then given by

$$u(t) = \int_a^b G(t, \big(qs + (1-q)a\big)\, z(s)Q(s)\,{}_a d_q s, \tag{10}$$

$$G(t,s) = g(t,s) + \frac{Ag(\epsilon, s)(t-a)^{\alpha-1}}{(b-a)^{\alpha-1} - A(\epsilon - a)^{\alpha-1}}, \tag{11}$$

where

$$\Gamma_q(\alpha) g(t,s) = \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} (b-a)^{\alpha-1}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, \\ \quad a \leq s \leq t, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} (b-s)^{\alpha-1}, & t \leq s \leq b. \end{cases} \tag{12}$$

Proof. We apply Lemma 2.3 in order to reduce the fractional boundary value problem (4) to an equivalent integral one

$$u(t) = -{}_a I_q^\alpha\, u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2}, \tag{13}$$

where c_1, c_2 are real constants.

From $u(a) = 0$ and (4), we get $c_2 = 0$. Therefore, the general solution of (4) is given by

$$u(t) = -{}_a I_q^\alpha\, u(t) + c_1(t-a)^{\alpha-1}$$

$$= -\int_a^t \frac{(t-(qs+(1-q)a))_a^{(\alpha-1)}}{\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$+c_1(t-a)^{\alpha-1}.$$

From (14), we deduce that

$$u(b) = -\int_a^b \frac{(b-(qs+(1-q)a))_a^{(\alpha-1)}}{\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$+ c_1(b-a)^{\alpha-1}, \tag{14}$$

and

$$u(\epsilon) = -\int_a^\epsilon \frac{(\epsilon-(qs+(1-q)a))_a^{(\alpha-1)}}{\Gamma_q(\alpha)} Q(s)z(s)_a d_q s,$$
$$+ c_1(\epsilon-a)^{\alpha-1}. \tag{15}$$

Now, the boundary condition $u(b) = Au(\epsilon)$ yields

$$c_1 = \int_a^b \frac{(b-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)u(s)_a d_q s \tag{16}$$
$$- \int_a^\epsilon \frac{A(\epsilon-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)u(s)_a d_q s, \tag{17}$$

where

$$\gamma := (b-a)^{\alpha-1} - A(\epsilon-a)^{\alpha-1}.$$

Thus, the non-trivial solution of (4) is uniquely given by

$$u(t) = -\int_a^t \frac{(t-(qs+(1-q)a))_a^{(\alpha-1)}}{\Gamma_q(\alpha)} Q(s)z(s)_a d_q s \tag{18}$$
$$+ \int_a^b \frac{(t-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$- \int_a^\epsilon \frac{A(t-a)^{\alpha-1}(\epsilon-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$= -\int_a^t \frac{(t-(qs+(1-q)a))_a^{(\alpha-1)}}{\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$+ \int_a^b \frac{(t-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)z(s)_a d_q s,$$
$$+ \int_a^b \frac{A(t-a)^{\alpha-1}(\epsilon-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$- \int_b^\epsilon \frac{A(t-a)^{\alpha-1}(\epsilon-(qs+(1-q)a))_a^{(\alpha-1)}}{\gamma\Gamma_q(\alpha)} Q(s)z(s)_a d_q s$$
$$= \int_a^b G(t,s)Q(s)z(s)_a d_q s,$$

where the Green function G is defined in (11) and (12), and the proof is finished.

Lemma 2.5. *Let $u \in AC[a,b]$. The q fractional boundary value problem*

$$\begin{cases} {}_aD_q^\beta(\phi_p({}_aD_q^\alpha u(t)) + Q(t)z(t) = 0, & t \in (a,b), \\ u(a) = 0, \quad u(b) = Au(\epsilon), \\ {}_aD_q^\alpha(a) = 0, \quad {}_aD_q^\alpha u(b) = B\,{}_aD_q^\alpha u(\delta),, \end{cases} \tag{19}$$

$1 < \alpha,\ \beta < 2,,\ a < \epsilon < b,$ *and* $0 \le A,\ B \le 1,$
then admits a non-trivial unique solution defined by

$$u(t) = \int_a^b G(t,(qs+(1-q)a))\,\phi_r \tag{20}$$
$$\left(\int_a^b H(s,(q\tau+(1-q)a))z(\tau)Q(\tau)d_q\tau\right) d_q s,$$

where $G(t,s)$ *is defined in* (11), (12) *and*

$$H(t,s) := h(t,s) + \frac{(B)^{p-1}h(\delta,s)(t-a)^{\alpha-1}}{(b-a)^{\beta-1}-(\delta-a)^{\beta-1}}, \tag{21}$$

where

$$\Gamma_q(\alpha)h(t,s) = \begin{cases} \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-s)^{\beta-1} - (t-s)^{\beta-1}, & a \le s \le t, \\ \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-s)^{\beta-1}, & t \le s \le b. \end{cases} \tag{22}$$

Proof. We use Lemma 2.4 in order to reduce the fractional differential Equation (4) to an equivalent integral one

$$\phi_p({}_aD_q^\alpha u(t)) = ({}_aD_q^\beta u(t)) + c_3(t-a)^{\beta-1} + c_4(t-a)^{\beta-2}. \tag{23}$$

In view of the boundary condition ${}_aD_q u(a) = 0$ and (23), we obtain $c_4 = 0$. Hence the non-trivial solution of the fractional boundary value (4) is given by

$$\phi_p({}_aD_q^\alpha u(t)) = ({}_aD_q^\beta u(t)) + c_3(t-a)^{\beta-1} \tag{24}$$
$$= \int_a^t \frac{(t-(qs+(1-q)a)_a)^{(\beta-1)}}{\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s$$
$$+ c_3(t-a)^{\beta-1}.$$

Now in light of (24), we get

$$\phi_p({}_aD_q^\alpha u(b)) = \int_a^b \frac{(b-(qs+(1-q)a))_a^{(\beta-1)}}{\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s$$
$$+ c_3(b-a)^{\beta-1}, \tag{25}$$
$$\phi_p({}_aD_q^\alpha u(\delta)) = \int_a^\delta \frac{(\delta-(qs+(1-q)a))_a^{(\beta-1)}}{\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s$$
$$+ c_3(\delta-a)^{\beta-1}. \tag{26}$$

By the boundary condition ${}_aD_q^\alpha u(b) = B\,{}_aD_q^\alpha u(\delta)$ yields

$$c_3 = \int_a^b \frac{(b-(qs+(1-q)a)_a)^{(\beta-1)}}{((b-a)^{\beta-1} - b^{p-1}(\delta-a)^{\beta-1})\,\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s$$

$$- \int_a^\delta \frac{(\delta - (qs + (1-q)a))_a^{(\beta-1)}}{((b-a)^{\beta-1} - b^{p-1}(\delta-a)^{\beta-1})\,\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s.$$

$$= \frac{1}{\bar{\gamma}} \left(\int_a^b \frac{(b - (qs + (1-q)a))_a^{(\beta-1)}}{\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s \right)$$
$$- \frac{1}{\bar{\gamma}} \left(\int_a^\delta \frac{(\delta - (qs + (1-q)a))_a^{(\beta-1)}}{\Gamma_q(\beta)} Q(s)z(s)\,{}_a d_q s \right), \quad (27)$$

where

$$\bar{\gamma} := ((b-a)^{\beta-1} - b^{p-1}(\delta-a)^{\beta-1}).$$

One may observe that, in a similar way to Lemma 2.4, we get

$$\phi_p({}_aD_q^\alpha u(t) = - \int_a^b H(t, (qs + (1-q)a))_a Q(s)z(s)\,{}_a d_q s. \quad (28)$$

Thus, the given fractional boundary value problem (4) may be re-written equivalently as

$$({}_aD_q^\alpha u(t) + \phi_r \left(\int_a^b H(t, (qs + (1-q)a))_a Q(s)u(s)\,{}_a d_q s \right)$$
$$= 0, \quad t \in (a,b),$$
$$u(a) = 0, \quad u(b) = Au(\delta).$$

Again by Lemma 2.4, the non-trivial solution of (4) is uniquely given by

$$u(t) = \int_a^b G(t, (qs + (1-q)a))\phi_r$$
$$\left(\int_a^b H(s, q\tau)z(\tau)Q(\tau)\,{}_a d_q \tau \right) {}_a d_q s. \quad (29)$$

The proof of the desired result is achieved.

Next we shall focus on finding the properties of the Green functions as well as their maximum principle. In order to do so, we express this fact in the following lemma.

Lemma 2.6. *Let $u \in C[a,b]$. The Green functions G and H defined respectively in* (11), (12) *and* (21), (22) *are then continuous and satisfy*

(a) $G(t, (qs + (1-q)a)) \geq 0$, *and* $H(t, (qs + (1-q)a)_a) \geq 0$, $\forall\ (t,s) \in [a,b] \times [a,b]$, (30)

(b) $G(t, qs + (1-q)a) \leq G(s, (qs + (1-q)a))$, *and* (31)
$H(t, qs + (1-q)a)) \leq H((qs + (1-q)a), (qs + (1-q)a))$
$\forall\ (t,s) \in [a,b] \times [a,b]$,

$1 < \alpha,\ \beta < 2,\ \ a < \epsilon < b$, *and* $0 \leq A,\ B \leq 1$.

Proof. Before starting the proof of Lemma 2.6, let us mention that γ and $\bar{\gamma}$ are positive, since $a < \epsilon,\ \delta < b$, and $0 \leq A,\ B \leq 1$.

We consider

$$G(t,s) = g(t,s) + \frac{Ag(\epsilon,s)(t-a)^{\alpha-1}}{(b-a)^{\alpha-1} - A(\epsilon-a)^{\alpha-1}}. \quad (32)$$

Let us differentiate $g(t,s)$ defined in (12) with respect to t, for $s \leq t$, by

$$\Gamma_q(\alpha)g(t,s) = \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, & t \leq s \leq b. \end{cases} \quad (33)$$

$$t({}_aD_q g(t,s) = t({}_aD_q((t-a)^{\alpha-1})\frac{(b-s)_a^{(\alpha-1)}}{(b-a)^{\alpha-1}}$$
$$-t({}_aD_q((t-s)_a^{(\alpha-1)},$$
$$= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)} \left((b-s)_a^{(\alpha-1)}(t-a)^{\alpha-2} \right)$$
$$-(t-s)_a^{(\alpha-2)} \left((t-a)^{\alpha-2} \right)$$
$$= \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(t-a)^{\alpha-2} \left((1 - \frac{s-a}{b-a})_0^{\alpha-1} \right)$$
$$-(t-a)^{\alpha-2} \left((1 - \frac{s-a}{t-a})_0^{\alpha-2} \right)$$
$$\leq \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(t-a)^{\alpha-2} \left((1 - \frac{s-a}{b-a})_0^{\alpha-1} \right)$$
$$- \left((1 - \frac{s-a}{b-a})_0^{\alpha-2} \right), \quad (34)$$

which is non-positive, since $a < s < t < b$.
Therefore, the function g is decreasing in its argument t, and the following inequality is satisfied

$$0 = g(b, qs + (1-q)a) \quad (35)$$
$$\leq g(t, qs + (1-q)a)$$
$$\leq g(qs + (1-q)a), qs + (1-q)a).$$

To this end, one may conclude that the right-hand-side of (35) may be expressed as

$$g(qs + (1-q)a), qs + (1-q)a) = \frac{1}{\Gamma_q(\alpha)}(\frac{q(s-a)}{b-a})^{\alpha-1}$$
$$\left(b - ((qs + (1-q)a)_a\right)_a^{(\alpha-1)}. (36)$$

Thus, $G(t, ((qs + (1-q)a))_a)$ is non-negative and satisfies

$$G(t, ((qs + (1-q)a))_a) \leq \max_{a \leq t \leq b} G(t, (qs + (1-q)a))$$
$$= \max_{a \leq t \leq b} (g(t, (qs + (1-q)a))$$
$$+ \frac{Ag(\epsilon, (qs + (1-q)a)(t-a)^{\alpha-1}}{\gamma})$$
$$\leq \frac{1}{\Gamma_q(\alpha)} \frac{q(s-a)^{(\alpha-1)}}{(b-a)^{\alpha-1}} \left(b - ((qs + (1-q)a)\right)_a^{(\alpha-1)}$$
$$+ \frac{Ag(\epsilon, (qs + (1-q)a)_a(b-a)^{\alpha-1}}{\gamma} := \tilde{G}_q(s). \quad (37)$$

For $t \leq s$, G is defined by

$$G(t,s) = g(t,s) + \frac{Ag(\epsilon,s)(t-a)^{\alpha-1}}{(b-a)^{\alpha-1} - A(\epsilon-a)^{\alpha-1}}, \quad (38)$$

where

$$\Gamma_q(\alpha)g(t,s) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}. \tag{39}$$

Similarly to above, we make differentiation with respect to t, and then we get

$$_t({}_aD_q g(t,s)) = \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\left(1-\frac{s-a}{b-a}\right)_a^{\alpha-1}(t-a)^{\alpha-2}\right), \tag{40}$$

which is non-negative, and consequently the function g is non-decreasing in its argument t. We have

$$\begin{aligned} 0 &= g(a, qs+(1-q)a) \\ &\leq g(t, qs+(1-q)a) \\ &\leq g(s, qs+(1-q)a), \end{aligned} \tag{41}$$

where $g(s, qs+(1-q)a) = \frac{(q(s-a))^{(\alpha-1)}}{(b-a)^{\alpha-1}}\big(b-((qs+(1-q)a)\big)_a^{(\alpha-1)}$.

Now, to prove the inequality involving H, we consider $H(t,s)$ defined in $(21)-(22)$ by

$$H(t,s) := h(t,s) + \frac{(B)^{p-1}h(\delta,s)(t-a)^{\beta-1}}{(b-a)^{\beta-1}-(B)^{p-1}(\delta-a)^{\beta-1}}, \tag{42}$$

$$\Gamma_q(\alpha)h(t,s) = \begin{cases} \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-s)^{\beta-1}-(t-s)^{\beta-1}, & a \leq s \leq t, \\ \\ \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-s)^{\beta-1}, & t \leq s \leq b. \end{cases} \tag{43}$$

For $t \leq s$, we have

$$\begin{aligned} \Gamma_q(\alpha)h(t,(qs+(1-q)a)_a &= \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-(qs+(1-q)a))_a^{(\beta-1)} \\ &\leq \frac{(s-a)^{(\beta-1)}}{(b-a)^{\beta-1}}(b-(qs+(1-q)a))_a{}^{(\beta-1)} \\ &:= \Gamma_q(\alpha)\, h(s,(qs+((1-q)a)). \end{aligned} \tag{44}$$

For $t \geq s$, we consider

$$\begin{aligned} \Gamma_q(\alpha)h(t,(qs+(1-q)a)_a) &= \frac{(t-a)^{\beta-1}}{(b-a)^{\beta-1}}(b-(qs+(1-q)a))_a{}^{(\beta-1)} \\ &- (t-(qs+(1-q)a))_a{}^{(\beta-1)}. \end{aligned} \tag{45}$$

We claim that $h(t,(qs+(1-q)a)_a)$ is non-negative too. It is sufficient to replace $\alpha-1$ by $\beta-1$ in all the steps of the proof of Lemma 2.6 (b), and we get the same result. So the proof is omitted, since it is similar to that of Lemma 2.6. Therefore

$$\begin{aligned} 0 &= h(a,(qs+(1-q)a)) \\ &\leq h(t,(qs+(1-q)a)) \\ &\leq h(s,(qs+(1-q)a)). \end{aligned}$$

Likely, one may conclude that the right-hand-side $h(s,(qs+(1-q)a))$ appearing in the previous inequality may be expressed as

$$\begin{aligned} h(s,(qs+(1-q)a)) = \frac{1}{\Gamma_q(\beta)}\left(\frac{q(s-a)}{b-a}\right)^{\beta-1} \\ \big(b-(qs+(1-q)a\big)_a^{(\beta-1)}. \end{aligned} \tag{46}$$

Thus, $H(t,(qs+(1-q)a))$ is non-negative and satisfies

$$\begin{aligned} H(t,(qs+(1-q)a)) &\leq \max_{a\leq t\leq b} H(t,(qs+(1-q)a)) \\ &= \max_{a\leq t\leq b}\Big(h(t,(qs+(1-q)a)) \\ &\quad + \frac{Ah(\delta,(qs+(1-q)a)_a(t-a)^{\beta-1}}{\bar{\gamma}}\Big) \\ &\leq \frac{1}{\Gamma_q(\beta)}\frac{q(s-a)^{(\beta-1)}}{(b-a)^{\beta-1}} \\ &\quad \big(b-((qs+(1-q)a)\big)_a^{(\beta-1)} \\ &+ \frac{Ah(\delta,(qs+(1-q)a)_a(b-a)^{\beta-1}}{\bar{\gamma}} \\ &:= \tilde{H}_q(s). \end{aligned} \tag{47}$$

The main result of this paper, which is a Lyapunov's inequality for a q- fractional difference p-Laplacian boundary value problem (4), will be formulated in the next theorem. We state and prove it in light of the previous lemmas.

Theorem 2.1. *Assume that u is a non-trivial solution of the q fractional boundary value problem*

$$\begin{cases} {}_aD_q^{\beta}(\phi_p({}_aD_q^{\alpha}\phi_p(u(t)))+Q(t)\phi_p(u(t))=0, & t\in(a,b), \\ u(a)=0, \quad u(b)=Au(\epsilon), \\ {}_aD_q^{\alpha}(a)=0, \quad {}_aD_q^{\alpha}u(b)=B\,{}_aD_q^{\alpha}u(\delta), \end{cases} \tag{48}$$

where ${}_aD_q^{\alpha}$, ${}_aD_q^{\beta}$ are the fractional q-derivative of the Riemann-Liouville type with $1<\alpha,\ \beta<2$, $0\leq A,\ B\leq 1$, $a<\epsilon,\ \delta<b$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_r$, $\frac{1}{p}+\frac{1}{r}=1$, and $Q:[a,b]\to R$ is a continuous function on $[a,b]$.

The following integral inequality is then satisfied

$$1 \leq \left(\int_a^b \tilde{G}_q(s)\,{}_ad_qs\right)\left(\int_a^b \tilde{H}_q(s)\,|Q(s)|\,{}_ad_qs\right)^{r-1}, \tag{49}$$

where $\tilde{G}_q(s)$, and $\tilde{H}_q(s)$ are defined in (37) and (49), respectively.

Proof. Let us define the norm of u, where u is a non-trivial solution of the q-fractional difference boundary value problem (4) by

$$||u|| := \max_{t\in[a,b]} |u(t)|.$$

Then, in view of Lemma 2.5, the non-trivial solution $u \in AC([a,b],\mathbb{R})$ may be re-written for all $t\in[a,b]$ as follows

$$u(t) = \int_a^b G(t, qs+(1-q)a)\phi_q \tag{50}$$

$$\left(\int_a^b H(s, q\sigma) + (1-q)a)\, Q(\sigma)\phi_p(u(\sigma))\, {}_a d_q\sigma\right) {}_a d_q s.$$

We then deduce

$$\begin{aligned} |u(t)| &\le \int_a^b |G(t, qs + (1-q)a))|\, |\phi_q\left(\int_a^b H(s,\sigma)\, Q(\sigma)\phi_p(u(\sigma))\, {}_a d_q\sigma\right)|\, {}_a d_q s \\ &= \int_a^b |G(t, qs+(1-q)a))| \\ &\quad |\left(\int_a^b H(s, q\sigma) + (1-q)a)\, Q(\sigma)\phi_p(u(\sigma))\, {}_a d_q\sigma\right)|^{r-1}\, {}_a d_q s \\ &= \int_a^b |G(t, qs+(1-q)a))| \\ &\quad |\left(\int_a^b |H(s, q\sigma) + (1-q)a)\, Q(\sigma)|\, ||u||_a^{p-1} d_q\sigma\right)|^{r-1}\, {}_a d_q s \\ &= \int_a^b |G(t,s)|||u||^{\frac{p-1}{r-1}} \left(\int_a^b |H(s, q\sigma) + (1-q)a)|\, |Q(\sigma)|\, {}_a d_q\sigma\right)^{r-1} {}_a d_q s. \end{aligned} \tag{51}$$

Based on the non-triviality of the solution u and the fact that p and r are conjugates, one may observe that

$$1 \le \int_a^b |G(t, qs+(1-q)a)|\, {}_a d_q s \left(\int_a^b |H(s, q\sigma) + (1-q)a)|\, |Q(\sigma)|\, {}_a d_q\sigma\right)^{r-1}.$$

Due to Lemma 2.5 and Lemma 2.6, it holds that

$$1 \le \int_a^b \tilde{G}_q(s)\, {}_a d_q s \left(\int_a^b \tilde{H}_q(s)\, |Q(\sigma)|\, {}_a d_q\sigma\right)^{r-1}.$$

To this end, it is worth noticing that, by letting q to 1^-, we retrieve the following integral inequality due to Hartman and Wintner (see [52])

$$\int_a^b (s-a)(b-s)|Q(s)|\, ds \ \ge\ (b-a).$$

Due to this fact, the obtained integral inequality (49) may be viewed as the q-fractional integral Hartman and Wintner inequality for the p-Laplacian case. However, it is easy for the reader to get an analogous result to this fundamental inequality by considering $p = 2, \alpha = 2$, and $q \to 1^-$.

Several types of Lyapunov's inequality were derived from Theorem 2.1. Hereafter, we formulate and express all of them in the following corollaries. We shall focus on covering both cases, ordinary differential equations and fractional differential equations. In addition, we illustrate this Theorem by giving an example. It consists of getting the interval of non-zeros of an appropriate eigenvalue fractional boundary value problem.

Let us start with the first result derived from Theorem 2.1.

Corollary 2.1. *Suppose that u is a non-trivial solution of the q fractional boundary value problem*

$$\begin{cases} {}_aD_q^\beta(\phi_p({}_aD_q^\alpha u(t)) + Q(t)\phi_p(u(t)) = 0, & t \in (a,b), \\ u(a) = 0, \quad u(b) = A\, u(\epsilon), \\ {}_aD_q^\alpha(a) = 0, \quad {}_aD_q^\alpha u(b) = B\, {}_aD_q^\alpha u(\delta),, \end{cases} \tag{52}$$

where ${}_aD_q^\alpha$, ${}_aD_q^\beta$ are the fractional q-derivative of Riemann-Liouville type with $1 < \alpha, \beta < 2$, $0 \le A, B \le 1$, $a < \epsilon, \delta < b$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_r$, $\frac{1}{p} + \frac{1}{r} = 1$, and $Q:[a,b] \to R$ is a continuous function on $[a,b]$.

The following integral inequality is then satisfied

$$1 \le \frac{1}{\Gamma_q(\alpha)} \frac{A}{\gamma}(\epsilon - a)^{(\alpha-1)} \frac{1}{\Gamma_q(\beta)} \frac{B^{p-1}}{\bar{\gamma}}(\delta - a)^{\beta-1}(b-a) \left(\int_a^b |Q(s)|\, {}_a d_q s\right)^{r-1}. \tag{53}$$

Proof. It is sufficient to let $q \to 0^+$ and consider two cases: $t \le s$ and $s \le t$. In the first case, $\tilde{G}_q(t,s)$ and $\tilde{H}_q(t,s)$ defined in (37) and (47) tend to zero. In the second case, they take the following form

$$\tilde{G}_q(t,s) := \frac{A(\epsilon - a)^{\alpha-1}}{\gamma} \text{ and } \tilde{H}_q(t,s) ::= \frac{B^{p-1}(\delta - a)^{\beta-1}}{\bar{\gamma}},$$

and we get the result of this corollary.

Corollary 2.2. *Suppose that u is a non-trivial solution of the q fractional boundary value problem*

$$\begin{cases} {}_aD_q^\beta(\phi_p({}_aD_q^\alpha u(t)) + Q(t)\phi_p(u(t)) = 0, & t \in (a,b), \\ u(a) = 0, \quad u(b) = A\, u(\epsilon), \\ {}_aD_q^\alpha(a) = 0, \quad {}_aD_q^\alpha u(b) = B\, {}_aD_q^\alpha u(\delta), \end{cases} \tag{54}$$

where ${}_aD_q^\alpha$, ${}_aD_q^\beta$ are the fractional q-derivative of the Riemann-Liouville type with $1 < \alpha, \beta < 2$, $0 \le A, B \le 1$, $a < \epsilon, \delta < b$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_r$, $\frac{1}{p} + \frac{1}{r} = 1$, and $Q:[a,b] \to R$ is a continuous function on $[a,b]$.

The following integral inequality is then satisfied

$$\begin{aligned} 1 \le &\left(\int_a^b \frac{1}{\Gamma_q(\alpha)}(\frac{s-a}{b-a})^{\alpha-1}(b-s)^{\alpha-1}\, {}_a d_q s + \int_a^b \frac{A}{\gamma} g(\epsilon, s)(b-a)^{\alpha-1}\, {}_a d_q s\right) \\ &\left(\int_a^b \frac{1}{\Gamma_q(\beta)}(\frac{s-a}{b-a})^{\beta-1}(b-s)^{\beta-1}\, |Q(s)|\, {}_a d_q s + \int_a^b \frac{B^{p-1}}{\bar{\gamma}} h(\delta, s)(b-a)^{\beta-1}\, |Q(s)|\, {}_a d_q s\right)^{r-1}. \end{aligned}$$

Proof. The result is achieved by letting $q \to 1^-$ in (49).

Remarks:

- The result of this corollary (Corollary 2.2) represents a Hartman-Wintner inequality for the q-fractional difference p-Laplacian boundary value problem (54). For the particular case when $A = B = 0$ and $\alpha = \beta$, we obtain

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}Q(s)ds \;\geq\; \Gamma(\alpha)(\frac{4}{b-a})^{\alpha-1}.$$

- When $\alpha = \beta = 2$, we retrieve the necessary condition of existence of non-trivial solutions investigated by Lyapunov for the second ordinary differential equation subject to the Dirichlet boundary conditions, and therefore one may conclude that if the non-trivial solution corresponding to this problem exists, then the non-trivial solution of (54) exists too, and vice-versa.

  Indeed, in that case, for $\alpha = \beta = 2$, we find:

$$\frac{4}{b-a} \;\leq\; \int_a^b (s-a)(b-s)\,|Q(s)|\,ds.$$

Now we focus on a second mixed-order differential inequality by taking $\alpha = \beta = 2$. For the next derived result from Theorem 2.1, we provide an important inequality that is very useful. This is the arithmetic-geometric-harmonic inequality. It says that:

$$(s-a)(b-s) \leq \frac{(b-a)^2}{4}.$$

Corollary 2.3. *Suppose that u is a non-trivial solution of the q fractional boundary value problem*

$$\begin{cases} {}_aD_q^{\beta}(\phi_p({}_aD_q^{\alpha}u(t)) \;+\; Q(t)\phi_p(u(t)) = 0, \quad t \in (a,b), \\ u(a) = 0, \quad u(b) = A\,u(\epsilon), \\ {}_aD_q^{\alpha}(a) = 0, \quad {}_aD_q^{\alpha}u(b) = B\,{}_aD_q^{\alpha}u(\delta), \end{cases} \tag{55}$$

where ${}_aD_q^{\alpha}$, ${}_aD_q^{\beta}$ *are the fractional q-derivative of the Riemann-Liouville type with* $1 < \alpha,\ \beta < 2$, $0 \leq A,\ B \leq 1$, $a < \epsilon,\ \delta < b$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_r$, $\frac{1}{p} + \frac{1}{r} = 1$, *and* $Q:[a,b] \to R$ *is a continuous function on* $[a,b]$.

The following integral inequality is then satisfied

$$\begin{aligned} 1 \;\leq\; &\left(\int_a^b \frac{1}{\Gamma(\alpha)}\frac{(b-a)^{2(\alpha-1)}}{4^{\alpha-1}}\,ds + \int_a^b \frac{A}{\gamma}g(\epsilon,s)(b-a)^{\alpha-1}\,ds\right) \\ &\left(\int_a^b \frac{1}{\Gamma(\beta)}(\frac{(b-a)^{2(\beta-1)}}{4^{\beta-1}}\,|Q(s)|\,ds \right.\\ &\left. + \int_a^b \frac{B^{p-1}}{\bar{\gamma}}h(\delta,s)(b-a)^{\beta-1}\,|Q(s)|\,ds\right)^{r-1}. \end{aligned}$$

Proof. We use the result of Corollary 2.2 by considering the arithmetic-geometric-harmonic inequality, and we get the desired result. Now we focus on a second mixed-order differential inequality by taking $\alpha = \beta = 2$, $p = 2$ and therefore $r = 2$, since p and r are conjugates.

Corollary 2.4. *Suppose that u is a non-trivial solution of the fractional q-difference boundary value problem*

$$\begin{cases} {}_aD''({}_aD''u(t) \;+\; Q(t)u(t) = 0, \quad t \in (a,b), \\ u(a) = 0, \quad u(b) = A\,u(\epsilon), \\ {}_aD''(a) = 0, \quad {}_aD''u(b) = B\,{}_aD''u(\delta), \end{cases} \tag{56}$$

where ${}_aD''$, ${}_aD''$ *are the fractional derivative of the Riemann-Liouville type of order* 2, $0 \leq A,\ B \leq 1$, $a < \epsilon,\ \delta < b$, *and* $Q:[a,b] \to R$ *is a continuous function on* $[a,b]$.

The following integral inequality is then satisfied

$$\begin{aligned} 1 \;\leq\; &\left(\int_a^b (\frac{s-a}{b-a})(b-s)\,ds + \int_a^b \frac{A}{\gamma}g(\epsilon,s)(b-a)\,ds\right) \\ &\left(\int_a^b (\frac{s-a}{b-a})(b-s)\,|Q(s)|\,ds + \int_a^b \frac{B}{\bar{\gamma}}h(\delta,s)(b-a)\,|Q(s)|\,ds\right), \end{aligned}$$

where g and h are defined in (12) *and* (22), *respectively (with* $\alpha = \beta = 2$*), and* γ *and* $\bar{\gamma}$ *are defined by*

$$\gamma := (b-a) - A(\epsilon - a), \;\; and \;\; \bar{\gamma} := (b-a) - B^{p-1}(\delta - a).$$

Proof. We set $\alpha = \beta = 2$, $p = r = 2$, and we let $q \to 1^-$ in Corollary 2.2, and the desired result is therefore established.

Remark:

The result obtained in Corollary 2.4 is more general than the Hartman Wintner inequality. For the particular case when $A = B = 0$, we get the classical Hartman-Wintner inequality.

Corollary 2.5. *Suppose that u is a non-trivial solution of the fractional boundary value problem*

$$\begin{cases} {}_aD''({}_aD''u(t) \;+\; Q(t)u(t) = 0, \quad t \in (a,b), \\ u(a) = 0, \quad u(b) = 0, \\ {}_aD''u(a) = 0, \quad {}_aD''u(b) = 0, \end{cases} \tag{57}$$

where ${}_aD''$, ${}_aD''$ *are the fractional derivative of the Riemann-Liouville type of order* 2, *and* $Q:[a,b] \to \mathbb{R}$ *is a continuous function on* $[a,b]$.

The following integral inequality is then satisfied

$$\frac{4}{b-a} \leq \int_a^b (s-a)(b-s)|Q(s)|\,ds. \tag{58}$$

Proof. It is sufficient to use the arithmetic-geometric-harmonic inequality to the conclusion of Corollary 2.4 and set $A = B = 0$, and the result will follow.

Corollary 2.6. *Suppose that u is a non-trivial solution of the fractional boundary value problem*

$$\begin{cases} {}_aD''({}_aD''u(t) \;+\; Q(t)u(t) = 0, \quad t \in (a,b), \\ u(a) = 0, \quad u(b) = 0, \\ {}_aD''(a) = 0, \quad {}_aD''u(b) = 0, \end{cases} \tag{59}$$

where ${}_aD''$, ${}_aD''$ *are the fractional derivative of the Riemann-Liouville type of order* 2, *and* $Q:[a,b] \to R$ *is a continuous function on* $[a,b]$.

The following integral inequality is then satisfied

$$\left(\frac{4}{b-a}\right)^2 \leq \int_a^b |Q(s)|ds. \tag{60}$$

Proof. Similarly to the above, we apply two times the arithmetic-geometric-harmonic inequality to the result of Corollary 2.4, and the desired result is achieved.

3. ON AN INTERVAL OF REAL ZEROS OF THE MITTAG-LEFFLER FUNCTION

In this section, we are interested in getting the interval of real zeros of the following Mittag-Leffler function [43]:

$$E_\alpha(\lambda) = \Sigma_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(k\alpha+\beta)}, \quad \lambda, \beta \in \mathbb{C}, \text{ and } Re(\alpha) > 0,$$

where C denotes the set of complex numbers, and $R(\alpha)$ is the real part of α. The key tool in proving this result consists of an appropriate integral inequality of the following fractional boundary value problem.

Theorem 3.1. *Let u be a non-trivial solution of*

$$\begin{aligned}&{}_0D^\alpha({}_0D^\alpha(u(t))) + \lambda u(t) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2,\\ &u(0) = 0, \ u(1) = 0,\end{aligned}$$

$${}_0D^\alpha u(0) = 0, \ {}_0D^\alpha u(1) = 0, \tag{61}$$

then $|\lambda| \geq (\Gamma(\alpha)4^{\alpha-1})^2$.

Proof. We apply Corollary 2.3 with $A = B = 0$, $\alpha = \beta$, $p = 2$, $r = 2$, and $a = 0$, $b = 1$. We obtain

$$|\lambda| \geq (\Gamma(\alpha)4^{\alpha-1})^2,$$

and the proof is completed.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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REFERENCES

1. Chai G. Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator. *Bound Value Probl*. (2012) **2012**:18. doi: 10.1186/1687-2770-2012-18
2. El-Shahed M, Hassan HA. Positive solutions of q-difference equation. *Proc Am Math Soc*. (2010) **138**:1733–8. doi: 10.1090/S0002-9939-09-10185-5
3. Ferreira RAC. Nontrivial solutions for fractional q-difference boundary value problems, *Electron J Qual Theory Differ Equat*. (2010) **70**:1–10. doi: 10.14232/ejqtde.2010.1.70
4. Ferreira RAC. Positive solutions for a class of boundary value problems with fractional q-differences. *Comput Math Appl*. (2011) **61**:367–73. doi: 10.1016/j.camwa.2010.11.012
5. Ferreira RAC. On a Lyapunov-type inequality the zeros of a certain Mittag-Leffler function. *J Math Anal Appl*. (2014) **412**:1058–63. doi: 10.1016/j.jmaa.2013.11.025
6. Liapunov AM. Problème général de la stabilityé du movement. *Ann Fac Sci Univ Toulouse*. (1907) **2**:203–407.
7. Liang S, Zhang J. Existence and uniqueness of positive solutions for three-point boundary value problem with fractional q-differences. *J Appl Math Comput*. (2012) **40**:277–88. doi: 10.1007/s12190-012-0551-2
8. Liu X, Jia M. Multiple solutions for fractional differential equations with nonlinear boundary conditions. *Comput Math Appl*. (2010) **59**:2880–6. doi: 10.1016/j.camwa.2010.02.005
9. Marin M, Nicaise S. Existence and stability results for thermoelastic dipolar bodies with double porosity. *Continuum Mech. Thermodyn*. (2016) **28**:1645–57. doi: 10.1007/s00161-016-0503-4
10. Marin M, Baleanu D, Vlase S. Effect of microtemperatures for micropolar thermoelastic bodies. *Struct Eng Mech*. (2017) **61**:381–7. doi: 10.12989/sem.2017.61.3.381
11. Marin M, Craciun EM. Uniqueness results for a boundary value problem in dipolar thermoelasticity to model composite materials. *Compos B Eng*. (2017) **126**:27–37. doi: 10.1016/j.compositesb.2017.05.063
12. Pachpatte BG. Lyapunov type integral inequalities for certain differential equations. *Georgian Math J*. (1997) **4**:139–48. doi: 10.1023/A:1022930116838
13. Podlubny I. *Fractional Differential Equations*. San Diego, CA: Academic Press (1999).
14. Rong J, Bai C. Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. *Adv Differ Equat*. (2015) **2015**:10. doi: 10.1186/s13662-015-0430-x
15. Ren T, Chen X. Positive solutions of fractional differential equation with p-Laplacian operator. *Abstr Appl Anal*. (2013) **2013**:789836. doi: 10.1155/2013/789836
16. Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives, Theory and Applications*. Yverdon: Gordon and Breach (1993).
17. Su Y, Li Q, Liu X. Existence criteria for positive solutions of p-Laplacian fractional differential equations with derivative terms. *Adv Differ Equat*. (2013) **2013**:119. doi: 10.1186/1687-1847-2013-119
18. Tariboon J, Ntouyas SK. Quantum integral inequalities on finite intervals. *J Inequal Appl*. (2014) **2014**:121. doi: 10.1186/1029-242X-2014-121
19. Tariboon J, Ntouyas SK, Agarwal P. New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations. *Adv Differ Equat*. (2015) **2015**:18. doi: 10.1186/s13662-014-0348-8
20. Wang J, Xiang H, Liu Z. Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian operator. *Abstr Appl Anal*. (2010) **2010**:971824. doi: 10.1155/2010/971824
21. Wu W, Zhou X. Eigenvalue of fractional differential equations with p-Laplacian operator. *Discrete Dyn Nat Soc*. (2013) **2013**:137890. doi: 10.1155/2013/137890
22. Yao S, Wang G, Li Z, Yu L. Positive solutions for three-point boundary value problem of fractional differential equation with p-Laplacian operator. *Discrete Dyn Nat Soc*. (2013) **2013**:376938. doi: 10.1155/2013/376938
23. Agarwal RP. Certain fractional q-integrals and q-derivatives. *Proc Camb Philos Soc*. (1969) **66**:365–70. doi: 10.1017/S0305004100045060
24. Annaby MH, Mansour ZS. *q-Fractional Calculus and Equations, Lecture Notes in Mathematics 2056*. Berlin: Springer-Verlag (2012).

25. Anastassiou GA. On right fractional calculus. *Chaos Solit Fract*. (2009) **42**:365–76. doi: 10.1016/j.chaos.2008.12.013
26. Al-Salam WA. Some fractional q-integrals and q-derivatives. *Proc Edinb Math Soc*. (1966/1967) **15**:135140. doi: 10.1017/S0013091500011469
27. Atici FM, Eloe PW. Fractional q-calculus on a time scale. *J Nonlinear Math Phys*. (2007) **14**:333–44. doi: 10.2991/jnmp.2007.14.3.4
28. Ernst T. *The History of q-Calculus a New Method*. UUDM Report 2000:16. Department of Mathematics, Uppsala University (2000).
29. Jleli M, Ragoub L, Samet B. On a Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition. *J Funct Spaces*. (2015) **2015**:468536. doi: 10.1155/2015/468536
30. Jackson FH. On q-functions and a certain difference operator. *Trans R Soc Edinb*. (1909) **46**:253–81. doi: 10.1017/S0080456800002751
31. Jackson FH. On q-definite integrals. *Q Pure J Appl Math*. (1910) **41**:193–203.
32. Pachpatte BG. *Mathematical Inequalities*. North Holland Mathematical Library. Elsevier (2005).
33. Rajkovic PM, Marinkovic SD, Stankovic MS. On q-analogues of Caputo derivative and Mittag-Leffler function. *Fract Calc Appl Anal*. (2007) **10**:359–73. Available online at: http://eudml.org/doc/11332
34. Aktas MF. Lyapunov-type inequalities for a certain class of n-dimensional quasilinear systems. *Electron J Differ Equat*. (2013) **67**:18.1. doi: 10.1016/j.camwa.2012.02.019
35. Cakmak D. Lyapunov-type integral inequalities for certain higher order differential equations. *Appl Math Comput*. (2010) **216**:368–73. doi: 10.1016/j.amc.2010.01.010
36. Cakmak D. On Lyapunov-type inequality for a class of nonlinear systems. *Math Inequal Appl*. (2013) **16**:101–8. doi: 10.7153/mia-16-07
37. Cheng SS. A discrete analogue of the inequality of Lyapunov. *Hokkaido Math J*. (1983) **12**:105–12. doi: 10.14492/hokmj/1381757783
38. El-Shahed M. Positive solutions for boundary value problem of nonlinear fractional differential equation. *Abstr Appl Anal*. (2007) **2007**:10368. doi: 10.1155/2007/10368
39. Ferreira RAC. A Lyapunov-type inequality for a fractional boundary value problem. *Fract Calc Appl Anal*. (2013) **16**:978–84. doi: 10.2478/s13540-013-0060-5
40. He X, Tang XH. Lyapunov-type inequalities for even order differential equations. *Commun Pure Appl Anal*. (2012) **11**:465–73. doi: 10.3934/cpaa.2012.11.465
41. Jleli M, Samet B. On a Lyapunov-type inequality for a fractional differential equation with a mixed boundary condition. *Math Inequal Appl*. (2015) **18**:443–51. doi: 10.7153/mia-18-33
42. Jleli M, Samet B. A Lyapunov-type inequality for a fractional q-difference boundary value problem. *J Nonlinear Sci Appl*. (2016) **9**:1965–76. doi: 10.22436/jnsa.009.05.03
43. Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations, Vol. 204 of North-Holland Mathematics Studies*. Amsterdam: Elsevier (2006).
44. Kwong MK. On Lyapunov's inequality for disfocality. *J Math Anal Appl*. (1981) **83**:486–94. doi: 10.1016/0022-247X(81)90137-2
45. Pachpatte BG. On Lyapunov-type inequalities for certain higher order differential equations. *J Math Anal Appl*. (1995) **195**:527–36. doi: 10.1006/jmaa.1995.1372
46. Ma J, Yang J. Existence of solutions for multi-point boundary value problem of fractional q-difference equation. *Electron J Qual Theory Differ Equat*. (2011) **92**:1–10. doi: 10.14232/ejqtde.2011.1.92
47. El-Shahed M, Al-Askar FMA. Multiple positive solutions for nonlinear second boundary value problems. *Int Math Forum*. (2010) **5**:769–74.
48. Graef JR, Kong L, Kong Q. Application of the mixed monotone operator method to fractional boundary value problems. *Fract Differ Calc*. (2012) **2**:87–98. doi: 10.7153/fdc-02-06
49. Eliason SB. A Lyapunov inequality for a certain nonlinear differential equation. *J Lond Math Soc*. (1970) **2**:524–7. doi: 10.1112/jlms/2.Part_3.461
50. Parhi N, Panigrahi S. On Liapunov-type inequality for third-order differential equations. *J Math Anal Appl*. (1999) **233**:445–60. doi: 10.1006/jmaa.19 99.6265
51. Liu Y, Lu L. A class of fractional p-Laplacian integro-differential equations in Banach spaces. *Abstr Appl Anal*. (2013) **2013**:398632. doi: 10.1155/2013/398632
52. Hartman P, Wintner A. On an oscillation criterion of Liapounoff. *Am J Math*. (1951) **73**:885–90. doi: 10.2307/2372122

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