Review Article

Rainer Schuhmann*

Description of aspheric surfaces

https://doi.org/10.1515/aot-2019-0011
Received January 18, 2019; accepted March 22, 2019; previously published online May 14, 2019

Abstract: Aspheric surfaces, in particular rotationally invariant surfaces, can be described according to the ISO standard 10110 Part 12 as sagitta functions of the surface coordinates. Usually, such functions are standardized as a combination of conic terms and power series or orthogonal polynomials. Similar functions are applied for surface forms, which are not rotationally invariant as cylindric and toric surfaces. In the following, different forms of describing aspheric surfaces as given in the standard as well as other forms will be presented and compared in an overview, and their special features will be discussed.

Keywords: aspheric surfaces; cylindric surfaces; optical surface description.

1 Introduction

For describing forms of optical surfaces, several mathematical functions can be applied. Standardized forms of description for rotationally invariant surfaces, as well as for other surfaces of certain kind of symmetries like cylindric and toric surfaces, are determined in ISO 10110 'Optics and photonics – Preparation of drawings for optical elements and systems – Part 12: Aspheric surfaces' [1], which is under regular revisions. Various alternative equations for aspheric surfaces have been proposed within the last decades; some of them where implemented in optical design software, others have been modified and some of them became part of the ISO standard, partially presented in [2]. Depending on the type of aspheric surface and the function within an optical system, the different forms of description may be more or less useful. Surfaces of lower symmetry like off-axis aspheres up to free form or general surfaces have been standardized in ISO 10110 Part 19 'General description of surfaces and components' [3]. This overview is concentrated on surfaces considered in ISO 10110 Part 12 and will show other modified forms and their special properties. Furthermore, sets of relevant equations will be provided clearly arranged and in an entire structure using a continuous and uniform nomenclature.

***Corresponding author: Rainer Schuhmann,** MSE Metrology and Software Development, Berliner Glas KGaA Herbert Kubatz GmbH & Co., Waldkraiburger Str. 5, 10823 Berlin, Germany, e-mail: rainer.schuhmann@berlinerglas.de

2 Coordinate system and sign convention

As a basis for the following explanations, a standardized right-handed, orthogonal coordinate system x-y-z is used as shown in Figure 1. For on-axis surfaces, the z axis is the optical axis. The radius of the curvature is positive if the center of curvature is to the right of the vertex, and negative if the center of the curvature is to the left of the vertex. The sagitta of a surface point is positive if the point is to the right of the vertex, and negative if the point is to the left of the vertex.

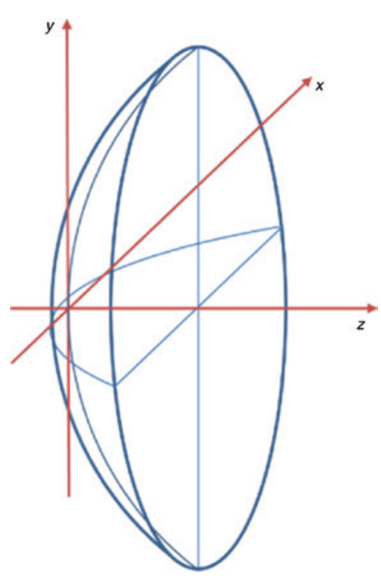

Figure 1: Surface coordinate system.

www.degruyter.com/aot

3 Surface descriptions

Within the coordinate system, a rotationally invariant optical surface can be described one-dimensionally by the surface sagitta value in the z direction as a function of the surface height

$$h=\sqrt{x^2+y^2} \tag{1}$$

as

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+f(h) \tag{2}$$

by a combination of two terms. With the first term, the basic conical surface will be described by its curvature ρ as the reciprocal of the radius of curvature r_0 and a conic constant κ. For $\kappa=0$, the basis is a spherical surface, and for $\rho=0$, the basis is plano. Figure 2 shows the different types of conic surfaces corresponding to the values of κ.

3.1 Surface descriptions by conic part and power series

The basis term is followed by a series expansion as a function of the surface height $f(h)$. Figure 3 shows the sagitta functions as defined in (2) for a basic sphere, given by the surface radius of curvature r_0, the basic conic section for $\kappa\neq 0$ and the aspheric surface of higher order. As a series expansion, a power series (monomials) can be used which adds monotone sagitta parts as deviations from the basic conic surface defined by the aspheric coefficients A_n as

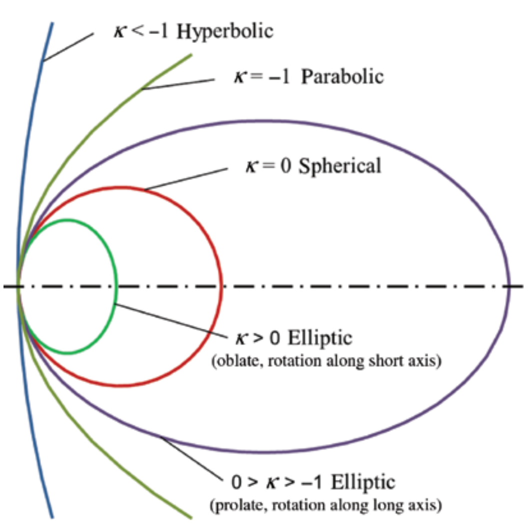

Figure 2: Conic surface forms.

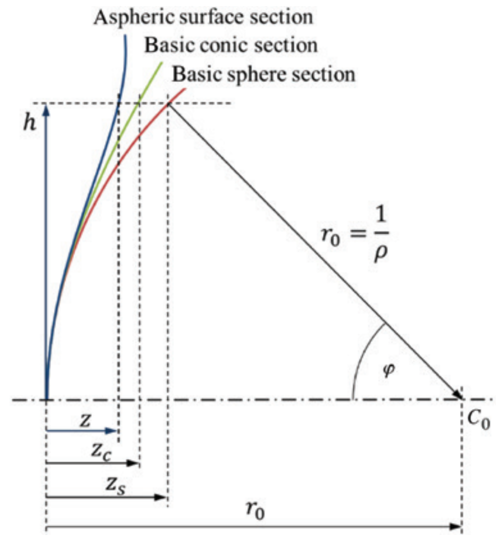

Figure 3: Surface sagitta form as a function of surface height.

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\sum_{n=1}^{N}A_n h^n \tag{3}$$

In the most standard way for such aspheric surfaces of higher orders, only even powers with A_{2n} starting from $n=2$ are considered only due to rotational invariance, so that

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\sum_{n=2}^{N}A_{2n} h^{2n} \tag{4}$$

This formula is ISO standardized for many years and used in several optical design softwares as well as a standard description on optical drawings. The disadvantages of this form are that the coefficients have different units of the order of (length)$^{1-2n}$, they normally have very different orders of magnitude and they do not remain constant if the surface is scaled.

An elegant way to overcome these disadvantages is, instead of describing the sagitta values as a function of the surface height, expanding the power series in the surface aperture with the surface aperture angle φ:

$$h\rho=\sin\varphi \tag{5}$$

so that a normalized power series is added to the basic term

$$z(h\rho)=\left[\frac{(h\rho)^2}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\sum_{n=2}^{N}B_{2n}(h\rho)^{2n}\right]\frac{1}{\rho} \tag{6a}$$

or

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\frac{1}{\rho}\sum_{n=2}^{N}B_{2n}(h\rho)^{2n} \tag{6b}$$

The standard power series (4) and the normalized power series (6) can be converted easily from and to each other by

$$B_{2n} = A_{2n}\rho^{1-2n} \tag{7}$$

The coefficients B_{2n} of this normalized power series are now free from units and have normally much lower differences in order of magnitude, and they remain constant while scaling, so that aspheric surfaces described in that way are much better comparable, independent from the surface focal length. This form is only limited to non-planar base surfaces with $\rho \neq 0$, which is not a general disadvantage. The application of this type of power series was proposed and published a long time ago (see [4–6]) and was implemented in optical design software at the Optical Institute of the Technical University Berlin and in the commercial software *WinLens* [7].

3.2 Surface descriptions by conic part and Zernike polynomials

In both power series versions, monotone interdependent series parts are applied to describe the sagitta deviation from the basic conic surface. This means that also the power coefficients are interdependent. Alternatively, to describe the deviation from a basic conic surface, orthogonal polynomials can be used, which was also proposed and published in [4–6] and implemented in optical design software long ago. Here, Zernike polynomials are used, which are defined as

$$Z_v^w(u, \varphi) = R_v^w(u)\cos(w\varphi) \tag{8}$$

for which $v>0$, $w>0$, $v \geq w$ and $(v-w)$ is even, u is the radial and φ is the azimuthal coordinate. Because of rotational invariance, for the surface description, only the Radial Zernike polynomials $R_v^w(u)$ are used, which are defined as

$$R_v^w(u) = \sum_{s=0}^{\frac{v-w}{2}} (-1)^s \frac{(v-s)!}{s!\left(\frac{v+w}{2}-s\right)!\left(\frac{v-w}{2}-s\right)!} u^{v-2s} \tag{9}$$

For rotationally invariant surfaces, even powers of the radial polynomial part of grade $w=4$ was selected with the variable $v=2n$, which results in

$$R_{2n}^4(u) = \sum_{s=0}^{n-2} (-1)^s \frac{(2n-s)!}{s!(n+2-s)!(n-2-s)!} u^{2(n-s)} \tag{10}$$

As the Zernike polynomials are defined for the unit circle, the variable parameter u is the normalized surface height for which the maximum surface height has to be specified by

$$u = \frac{h\rho}{h_{\max}\rho} = \frac{h}{h_{\max}} \tag{11}$$

The selection of the orthogonal Zernike polynomials corresponds to the nomenclature standardized in ISO/Technical Report 14999-2 ‘Optics and photonics – Interferometric measurement of optical elements and optical systems – Part 2: Measurement and evaluation techniques’ [8] for the Zernike order Z and N as listed in Table 1.

Following the normalization on the surface aperture as in (6), the sagitta function of an aspheric surface using the Zernike expansion normalized to the surface aperture $h_{\max}\rho$ with respect to the maximum surface height $h_{\max}$ can now be written as

$$z(h\rho) = \left[\frac{(h\rho)^2}{1+\sqrt{1-(1+\kappa)(h\rho)^2}} + \sum_{n=2}^{N} C_{2n} R_{2n}^4(u)\right]\frac{1}{\rho} \tag{12a}$$

or

$$z(h) = \frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}} + \frac{1}{\rho}\sum_{n=2}^{N} C_{2n} R_{2n}^4(u) \tag{12b}$$

with C_{2n} as Zernike coefficients. Without normalization to maximum surface height, the Zernike expansion can also be written as

$$z(h) = \frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}} + \sum_{n=2}^{N} \bar{C}_{2n} R_{2n}^4(u) \tag{13}$$

with the Zernike coefficients

$$\bar{C}_{2n} = \frac{1}{\rho} C_{2n} \tag{14}$$

Table 1: Zernike polynomial orders according to ISO/TR 14999-2.

Z	N	v	w	Radial part of Zernike polynomial
16/17	8	4	4/−4	$R_4^4(u)$
27/28	10	6	4/−4	$R_6^4(u)$
40/41	12	8	4/−4	$R_8^4(u)$
55/56	14	10	4/−4	$R_{10}^4(u)$
72/73	16	12	4/−4	$R_{12}^4(u)$
91/92	18	14	4/−4	$R_{14}^4(u)$

which represent the aspheric polynomial part as the sagitta deviation from the conic section at the surface edge for

$$h = h_{\max} \rightarrow u = \frac{h}{h_{\max}} = 1 \rightarrow R_{2n}^4(u) = 1$$

The Zernike polynomials R_{2n}^4 can be computed simply using the short form (9) or for higher numerical stability, which is only necessary for extremely higher orders, by using recursion algorithms in the following way:

$$R_{2n}^4(u) = \frac{1}{a(n)}[(b(n)+c(n)(2u^2-1)) \\ R_{2n-1}^4(u) - d(n)R_{2n-2}^4(u)] \tag{15a}$$

starting with

$$R_4^4(u) = 1 \tag{15b}$$

and following for $n > 2$ with

$$a(n) = (n+2)(2n-2)(2n-4) \tag{15c}$$

$$b(n) = -32n + 16 \tag{15d}$$

$$c(n) = 2n(2n-1)(2n-2) \tag{15e}$$

$$d(n) = 2n(n+1)(2n-5) \tag{15f}$$

Applying (10) or (15) for the computation of the Zernike polynomials up to the order of $2n = 14$ results in

$$R_4^4(u) = u^4 \tag{16a}$$

$$R_6^4(u) = -5u^4 + 6u^6 \tag{16b}$$

$$R_8^4(u) = 15u^4 - 42u^6 + 28u^8 \tag{16c}$$

$$R_{10}^4(u) = -35u^4 + 168u^6 - 252u^8 + 120u^{10} \tag{16d}$$

$$R_{12}^4(u) = 70u^4 - 504u^6 + 1260u^8 - 1320u^{10} + 495u^{12} \tag{16e}$$

$$R_{14}^4(u) = -126u^4 + 1260u^6 - 4620u^8 + 7920u^{10} \\ -6435u^{12} + 2002u^{14} \tag{16f}$$

The main advantage of using the Zernike polynomials for the description of the deviation from the basic conic surface is their orthogonality and, therefore, their independence from each other. This series expansion is a summation of non-monotone-independent parts, which can be used especially for the optical design process as shown in [4–7]. In this way, a targeted correction of zonal aberrations is possible in a much better way than using the standard power series.

3.3 Surface descriptions by conic part and Q^{con}-polynomials

An equivalent orthogonal surface form description for the deviation from a basic conic surface is proposed by Forbes [9–12] using the so-called Q^{con}-polynomials as modified Zernike polynomials as

$$z(h) = \frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}} + u^4 \sum_{m=0}^{M} a_m Q_m^{\text{con}}(u) \tag{17}$$

With $n = m + 2$ the equivalence of both orthogonal polynomial forms is given by

$$Q_m^{\text{con}} u^4 = R_{2(m+2)}^4 = R_{2n}^4 \tag{18}$$

and

$$a_m = \frac{1}{\rho} C_{2(m+2)} = \frac{1}{\rho} C_{2n} = \bar{C}_{2n} \tag{19}$$

So, this means that the Q^{con}-polynomials are just Zernike polynomials for which the factor u^4 is separated in front of the expansion sum, while the coefficients a_m are the correlated Zernike coefficients.

The polynomial form according to (17) is represented in the ISO 10110 Part 12, in which the computation of the orthogonal Q^{con}-polynomials is defined only using recursion algorithms. They are equivalent to the algorithms used for Zernike polynomials shown in (15) and are listed here in a simplified form as

$$Q_m^{\text{con}}(u) = \frac{1}{a(m)}[(b(m)+c(m)(2u^2-1)) \\ Q_{m-1}^{\text{con}}(u) - d(m)Q_{m-2}^{\text{con}}(u)] \tag{20a}$$

starting with

$$Q_0^{\text{con}}(u) = 1 \tag{20b}$$

and following for $m > 0$ with

$$a(m) = 2m(m+4)(2m+2) \tag{20c}$$

$$b(m) = -32m - 48 \tag{20d}$$

$$c(m) = (2m+2)(2m+3)(2m+4) \tag{20e}$$

$$d(m) = 2(m-1)(m+3)(2m+4) \tag{20f}$$

Also, here, a much easier way for computing the Q^{con}-polynomials for moderate orders is given similar to (10) by

$$Q_m^{\text{con}}(u^2)=\sum_{s=0}^{m}(-1)^s \frac{(2m+4-s)!}{s!(m+4-s)!(m-s)!}u^{2(m-s)} \qquad (21)$$

For the sake of completeness and in accordance with (16) and ISO 10110 Part 12, the computation of the Q^{con}-polynomials up to the order of $m=5$ results in

$$Q_0^{\text{con}}(u)=1 \qquad (22a)$$

$$Q_1^{\text{con}}(u)=-5+6u^2 \qquad (22b)$$

$$Q_2^{\text{con}}(u)=15-42u^2+28u^4 \qquad (22c)$$

$$Q_3^{\text{con}}(u)=-35+168u^2-252u^4+120u^6 \qquad (22d)$$

$$Q_4^{\text{con}}(u)=70-504u^2+1260u^4-1320u^6+495u^8 \qquad (22e)$$

$$Q_5^{\text{con}}(u)=-126+1260u^2-4620u^4+7920u^6 -6435u^8+2002u^{10} \qquad (22f)$$

Furthermore, as for the Zernike polynomial form, the coefficients a_m of the Q^{con}-polynomials represent the aspheric polynomial part as the sagitta deviation from the conic section at the surface edge for

$$h=h_{\max} \rightarrow u=\frac{h}{h_{\max}}=1 \rightarrow Q_m^{\text{con}}(u)=1$$

3.4 Conversion of series expansions

An important fact is that the orthogonal polynomial forms shown here as Zernike polynomials (13) and equivalent Q^{con}-polynomials (17) can be directly converted to the standard power series given in (4) and (6) using coefficient comparison by

$$A_4=h_{\max}^{-4}\,(\bar{C}_4-5\bar{C}_6+15\bar{C}_8-35\bar{C}_{10}+70\bar{C}_{12}-126\bar{C}_{14}) \qquad (23a)$$

$$A_6=h_{\max}^{-6}\,(6\bar{C}_6-42\bar{C}_8+168\bar{C}_{10}-504\bar{C}_{12}+1260\bar{C}_{14}) \qquad (23b)$$

$$A_8=h_{\max}^{-8}\,(28\bar{C}_8-252\bar{C}_{10}+1260\bar{C}_{12}-4620\bar{C}_{14}) \qquad (23c)$$

$$A_{10}=h_{\max}^{-10}\,(120\bar{C}_{10}-1320\bar{C}_{12}+7920\bar{C}_{14}) \qquad (23d)$$

$$A_{12}=h_{\max}^{-12}\,(495\bar{C}_{12}-6435\bar{C}_{14}) \qquad (23e)$$

$$A_{14}=h_{\max}^{-14}\,2002\bar{C}_{14} \qquad (23f)$$

and vice versa

$$\bar{C}_4=a_0=A_4h_{\max}^4+\frac{5}{6}A_6h_{\max}^6+\frac{5}{7}A_8h_{\max}^8 +\frac{5}{8}A_{10}h_{\max}^{10}+\frac{5}{9}A_{12}h_{\max}^{12}+\frac{1}{2}A_{14}h_{\max}^{14} \qquad (24a)$$

$$\bar{C}_6=a_1=\frac{1}{6}A_6h_{\max}^6+\frac{1}{4}A_8h_{\max}^8+\frac{7}{24}A_{10}h_{\max}^{10} +\frac{14}{45}A_{12}h_{\max}^{12}+\frac{7}{22}A_{14}h_{\max}^{14} \qquad (24b)$$

$$\bar{C}_8=a_2=\frac{1}{28}A_8h_{\max}^8+\frac{3}{40}A_{10}h_{\max}^{10} +\frac{6}{55}A_{12}h_{\max}^{12}+\frac{3}{22}A_{14}h_{\max}^{14} \qquad (24c)$$

$$\bar{C}_{10}=a_3=\frac{1}{120}A_{10}h_{\max}^{10}+\frac{1}{45}A_{12}h_{\max}^{12}+\frac{1}{26}A_{14}h_{\max}^{14} \qquad (24d)$$

$$\bar{C}_{12}=a_4=\frac{1}{495}A_{12}h_{\max}^{12}+\frac{1}{154}A_{14}h_{\max}^{14} \qquad (24e)$$

$$\bar{C}_{14}=a_5=\frac{1}{2002}A_{14}h_{\max}^{14} \qquad (24f)$$

This shows that a standard surface description, which allows maximum comparability of aspheric surfaces, is possible by using the power series as standardized in ISO 10110 Part 12. After using an orthogonal expansion form for the optical design process, the power series coefficients can be converted easily from them for the optical surface drawing.

3.5 Surface descriptions by conic part and *Q*-polynomials for base fits

An additional surface form description for the deviation from a basic conic surface is also proposed by Forbes [9–11] using the so-called Q^{bfs}-polynomials. This form is also present in ISO 10110-12. The special features of this form might have advantages for the design, the manufacturing and the measurement of aspherical surfaces. Again, the description of the sagitta values of a surface splits into two terms, the basic conic term and a sum of polynomials, as

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\frac{u^2(1-u^2)}{\sqrt{1-(h\rho)^2}}\sum_{m=0}^{M}b_mQ_m^*(u) \qquad (25)$$

As modification to the ISO standard, the polynomial Q is initially indexed here with $*$, which implies the characteristic deviation from the base surface, which can be conic or plano, in a way that the deviation fits the base at the surface vertex and the center at $h=0$, respectively, and the surface edge given by h_{max}. Again, u is the normalized surface height defined in (11).

Using the cosine of the local surface aperture angle φ as defined in (5) as a reciprocal factor in the series expansion of (25) as

$$\cos\varphi=\cos[\arcsin(h\rho)]=\sqrt{1-\sin^2\varphi}=\sqrt{1-(h\rho)^2} \tag{26}$$

the sagitta function can be written as

$$z(h)=\frac{h^2\rho}{1+\sqrt{1-(1+\kappa)(h\rho)^2}}+\frac{1}{\cos(\varphi)}f_\perp(h) \tag{27}$$

so that in a first approximation, the remaining function

$$f_\perp(h)=u^2(1-u^2)\sum_{m=0}^{M}b_mQ_m^*(u) \tag{28}$$

can be interpreted as the surface deviation from the basic surface in normal direction. This may create advantages for surface manufacturing and measurement.

A further important feature of this description form is the orthogonality of the first derivatives of the polynomial term, which describe the slopes of the surface around the basic conic form.

Besides the explicit standardized option of applying this form also for a non-spherical base surface, the main usage is proposed for a spherical base which is declared as the 'best fit spherical' surface. This only means that, again, the expansion term just fits the spherical surfaces at the surface vertex and its edge, as for fitted conic or plano surfaces. So, for a 'best fit sphere' (*bfs*) base with the curvature ρ_{bfs} in this sense here, the equation (25) is changed to

$$z(h)=\frac{h^2\rho_{bfs}}{1+\sqrt{1-(h\rho_{bfs})^2}}+\frac{u^2(1-u^2)}{\sqrt{1-(h\rho_{bfs})^2}}\sum_{m=0}^{M}b_mQ_m^{bfs}(u) \tag{29}$$

From another point of view, a real best fit spherical surface generally is defined as a spherical surface with minimum deviation from the real aspheric surface described by the whole equation.

The computation of the polynomials Q^* or Q^{bfs} respectively, is following the more complex recursion formulas as

$$Q_m^*(u)=\frac{1}{l_{m+1}}P_{m+1}(u)-g_mQ_m^*(u)-k_{m-1}Q_{m-1}^*(u) \tag{30a}$$

with

$$P_{m+1}(u)=(2-4u^2)P_m(u)-P_{m-1}(u) \tag{30b}$$

$$k_{m-2}=\frac{m(1-m)}{2l_{m-2}} \tag{30c}$$

$$g_{m-1}=\frac{-(1+g_{m-2}k_{m-2})}{l_{m-1}} \tag{30d}$$

$$l_m=\sqrt{m(m+1)+3-g_{m-1}^2-k_{m-2}^2} \tag{30e}$$

which have to be solved for $m\geq 2$ using the starting values:

$$Q_0^*(u)=1 \tag{30f}$$

$$Q_1^*(u)=\frac{1}{\sqrt{19}}(13-16u^2) \tag{30g}$$

$$P_0(u)=2 \tag{30h}$$

$$P_1(u)=6-8u^2 \tag{30i}$$

$$g_0=\frac{1}{2} \tag{30j}$$

$$l_0=2 \tag{30k}$$

$$l_1=\frac{1}{2}\sqrt{19} \tag{30l}$$

The resulting polynomial functions up to the order $m=5$ can then be written as

$$Q_0^*(u)=1 \tag{31a}$$

$$Q_1^*(u)=\sqrt{\frac{1}{19}}(13-16u^2) \tag{31b}$$

$$Q_2^*(u)=\sqrt{\frac{2}{95}}(29-4u^2(25-19u^2)) \tag{31c}$$

$$Q_3^*(u)=\sqrt{\frac{2}{2545}}(207-4u^2(315-u^2(577-320u^2))) \tag{31d}$$

$$Q_4^*(u)=\frac{1}{3\sqrt{131831}}(7737-16u^2(4653-2u^2(7381-8u^2(1168-509u^2)))) \tag{31e}$$

$$Q_5^*(u)=\frac{1}{3\sqrt{6632213}}(66657-32u^2(28338-u^2(135325-8u^2(35884-u^2(34661-12432u^2))))) \tag{31f}$$

3.6 Comparison of functional expansions

A graphical presentation of the different expansion functions demonstrates the impact of the single series parts onto the surface form more clearly. Figure 4 shows the sagitta functions $z(h)$ of the different orders $2n=4$ to 14 or $m=0$ to 5, respectively, normalized to the surface height $h_{max}=1$. It can be seen and understood that as opposed to the monotone power series function, the orthogonal function of the Zernike polynomials or Q^{con}-polynomials, respectively, and the Q^{*}-polynomials, here shown in the axial direction and in the surface normal direction without the cosine factor, allow better direct impacts onto special surface zones. In addition, Figure 5 demonstrates the corresponding 3-dimensional function for rotationally invariant surfaces more colorfully.

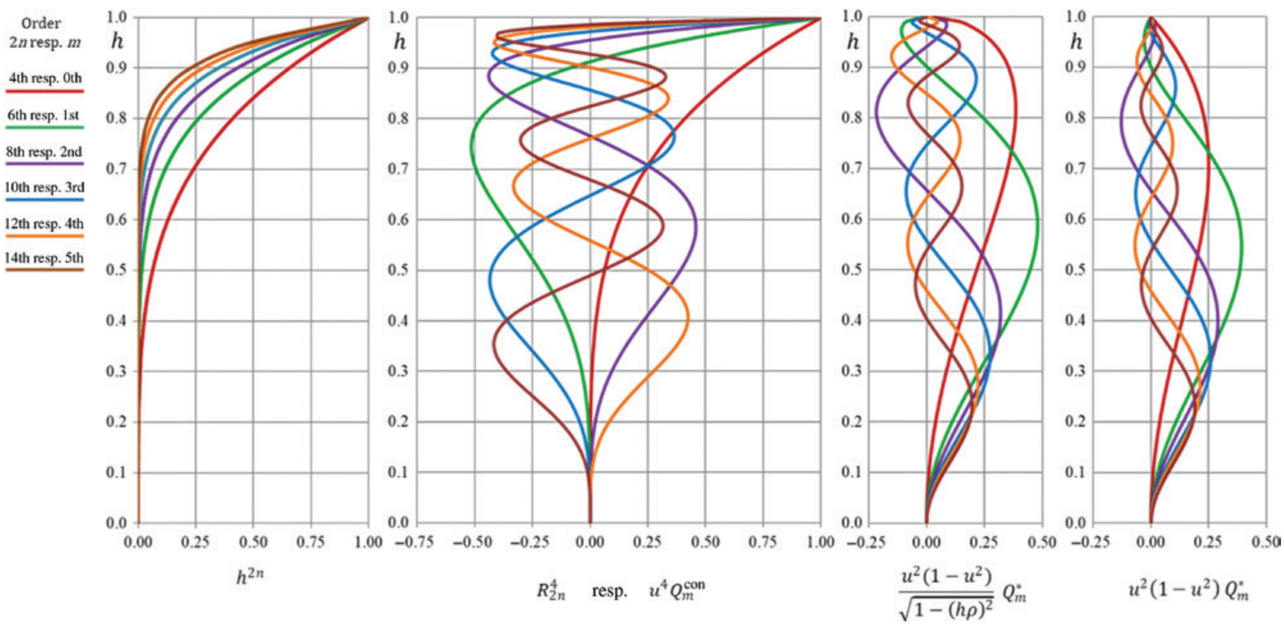

Figure 4: Surface sagitta form deviation from base for the orders $2n=4$ to 14 or $m=0$ to 5, respectively, as function of surface height for power series, Zernike polynomials and Q-polynomials for normalized surface height.

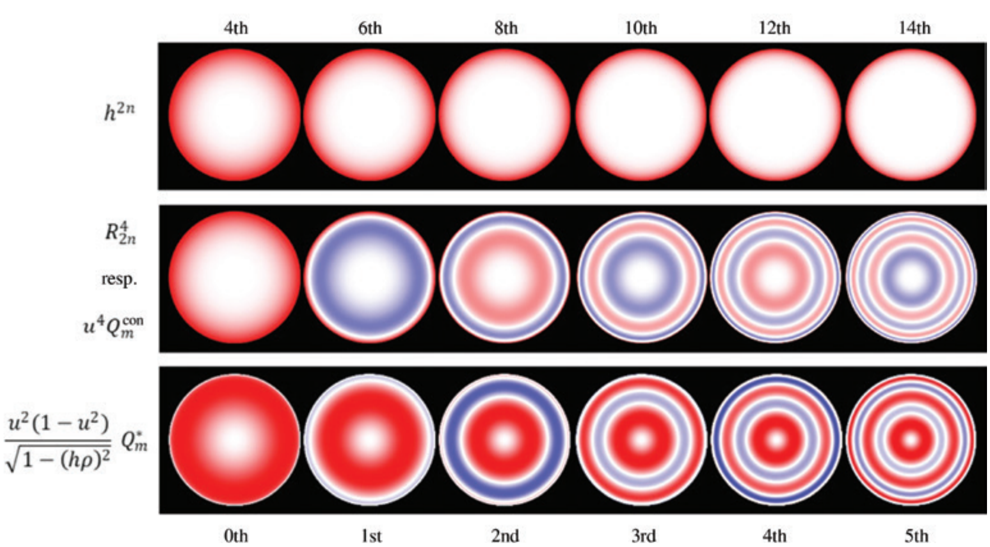

Figure 5: Two-dimensional surface sagitta form deviation from base for the orders $2n=4$ to 14 or $m=0$ to 5, respectively, for power series, Zernike polynomials and Q-polynomials for normalized surface height.

4 Slope calculation

Besides the general surface form, the variation of the surface slope as a function of the surface height is of special interest. The slope variation might be different compared to spherical or conic surfaces for which the slope is always monotone. The local slope as the surface gradient is the first partial derivative of the aspheric sagitta function to the surface height h. For the power series as defined in (4) and (6), the differentiation yields to

$$\frac{dz(h)}{dh}=\frac{h\rho}{\sqrt{1-(1+\kappa)(h\rho)^2}}+\sum_{n=2}^{N}2n\,A_{2n}\,h^{2n-1} \tag{32}$$

and

$$\frac{dz(h)}{dh}=\frac{h\rho}{\sqrt{1-(1+\kappa)(h\rho)^2}}+\sum_{n=2}^{N}2n\,B_{2n}\,(h\rho)^{2n-1} \tag{33}$$

For the sagitta functions, using the orthogonal Zernike (12) or Q^{con}-polynomials (17), respectively, the partial derivatives are

$$\frac{dz(h)}{dh}=\frac{h\rho}{\sqrt{1-(1+\kappa)(h\rho)^2}}+\frac{1}{h\rho}\begin{bmatrix} C_4\,4u^4 \\ +C_6(-20u^4+36u^6) \\ +C_8(60u^4-252u^6+224u^8) \\ +C_{10}(-140u^4+1008u^6-2016u^8+1200u^{10}) \\ +C_{12}(+280u^4-3024u^6+10080u^8-13200u^{10}+5940u^{12}) \\ +C_{14}(-504u^4+7560u^6-36960u^8+79200u^{10}-77220u^{12}+28028u^{14}) \end{bmatrix} \tag{34}$$

and

$$\frac{dz(h)}{dh}=\frac{h\rho}{\sqrt{1-(1+\kappa)(h\rho)^2}}+\frac{1}{h}\begin{bmatrix} a_0\,4u^4 \\ +a_1(-20u^4+36u^6) \\ +a_2(60u^4-252u^6+224u^8) \\ +a_3(-140u^4+1008u^6-2016u^8+1200u^{10}) \\ +a_4(+280u^4-3024u^6+10080u^8-13200u^{10}+5940u^{12}) \\ +a_5(-504u^4+7560u^6-36960u^8+79200u^{10}-77220u^{12}+28028u^{14}) \end{bmatrix} \tag{35}$$

As

$$\frac{dz(h)}{dh}=\frac{1}{h_{\max}}\frac{dz(u)}{du} \tag{36}$$

and using the substitution

$$t=\sin\varphi_{\max}=h_{\max}\rho \tag{37}$$

the differentiation of the sagitta function using the Q^*-polynomials (25) results in

$$
\begin{aligned}
\frac{dz(h)}{dh} &= \frac{h\rho}{\sqrt{1-(1+\kappa)(h\rho)^2}} \\
&+\frac{1}{h_{max}}\frac{1}{\sqrt{1-t^2u^2}^3}\left[\begin{aligned}
&b_1 u(u^2(t^2(3u^2-1)-4)+2) \\
&-b_2\sqrt{\frac{1}{19}}u(u^2(u^2(t^2(80u^2-87)-96)+13t^2+116)-26) \\
&+b_3\sqrt{\frac{2}{95}}u\left(u^2\left(u^2\left(4u^2\left(t^2\left(133u^2-220\right)-152\right)+387t^2+1056\right)-29t^2-516\right)+58\right) \\
&-b_4\sqrt{\frac{2}{2545}}u\left(u^2\left(u^2\left(4u^2\left(u^2\left(3t^2\left(960u^2-2093\right)-3200\right)+4460t^2+7176\right)-4401t^2-21408\right)+207t^2+5868\right)-414\right) \\
&+b_5\frac{1}{3}\sqrt{\frac{1}{131831}}u\left(u^2\left(u^2\left(32u^2\left(u^2\left(8u^2\left(t^2\left(5599u^2-15093\right)-6108\right)+117075t^2+134160\right)-48540t^2-133800\right)+246603t^2+1863936\right)-7737t^2-328804\right)+15474\right) \\
&-b_6\frac{1}{3}\sqrt{\frac{1}{6632213}}u\left(u^2\left(u^2\left(32u^2\left(u^2\left(8u^2\left(u^2\left(t^2\left(161616u^2-518023\right)-174048\right)+634905t^2+565116\right)-2956779t^2-5643600\right)+818315t^2+3379176\right)-2920419t^2-31423296\right)+66657t^2+3893892\right)-133314\right)
\end{aligned}\right]
\end{aligned}
\tag{38}
$$

Furthermore, the partial differentiation of the expansion function part $f_\perp(h)$ according to (28) determines the slope around the base surface in normal direction as

$$
\begin{aligned}
\frac{df_\perp(h)}{dh} &= \frac{d}{dh}\left[u^2(1-u^2)\sum_{m=0}^{M} b_m Q_m^{bfs}(u^2)\right] \\
&= \frac{1}{h_{max}}\cdot\left[\begin{aligned}
&b_1\,(2u-4u^3) \\
&+b_2\sqrt{\frac{1}{19}}\,(26u-116u^3+96u^5) \\
&+b_3\sqrt{\frac{2}{95}}\,(58u-516u^3+1056u^5-608u^7) \\
&+b_4\sqrt{\frac{2}{2545}}\,(414u-5868u^3+21408u^5-28704u^7+12800u^9) \\
&+b_5\sqrt{\frac{1}{131831}}(5158u-109580u^3+621280u^5-1427200u^7+1431040u^9-521216u^{11}) \\
&+b_6\sqrt{\frac{1}{6632213}}(44438u-1297964u^3+10474432u^5-36044544u^7+60198400u^9-48223232u^{11}+14852096u^{13})
\end{aligned}\right]
\end{aligned}
\tag{39}
$$

The partial derivatives for the power expansion, for the Zernike polynomial expansion, which is equivalent to the Q^{con}-expansion as well as for the Q^{*}-expansion in axial direction and in surface normal direction without the cosine factor are shown in Figure 6.

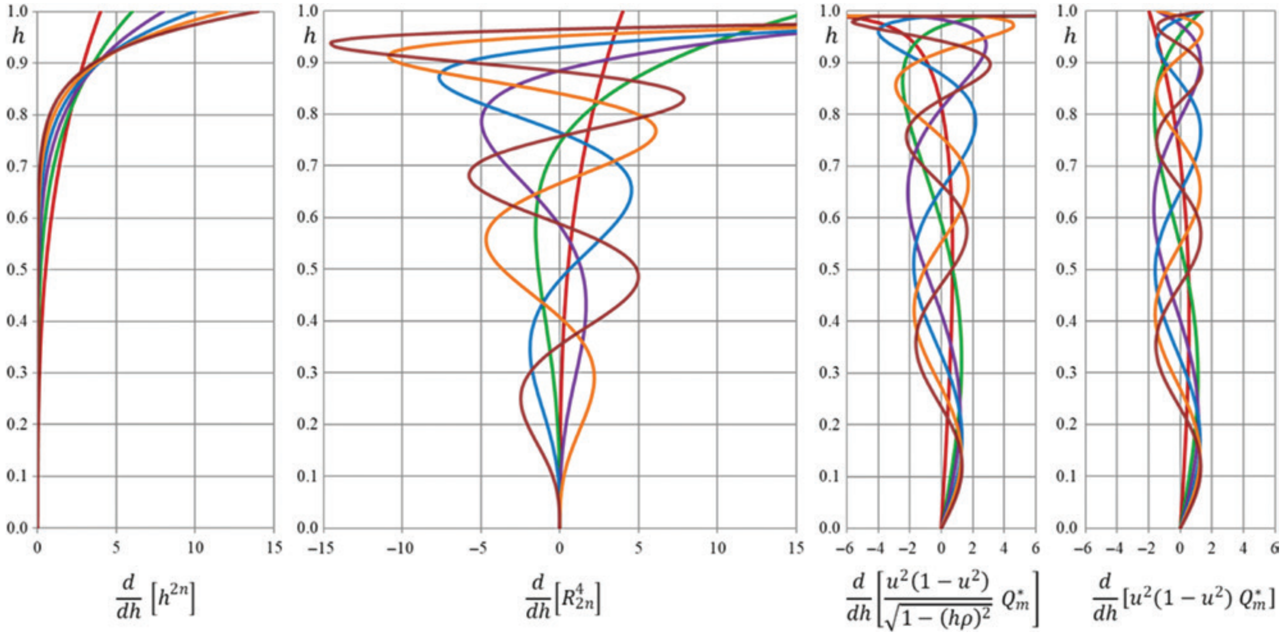

Figure 6: Slope variations as partial derivatives of power series, Zernike polynomials and Q-polynomials for normalized surface height (coloring of orders equivalent to Figure 4).

5 Scaling of aspheric surfaces

In the optimization process, optical surfaces are modified with respect to the surface form and surface aperture. This means that, for example, a surface will be scaled by a factor F_s, which changes the conic base with the radius of curvature and therefore the surface focal length as well as the maximum surface height. To keep the characteristic aspheric form of such a scaled surface means that the sagitta function $z(h)$ will be scaled and that it might be necessary to change the aspheric coefficients of the series expansion by an equivalent scaling also, depending on the type of that expansion. Furthermore, in technical drawings of lenses, the orientation and, therefore, the sign of the radius of curvature is dependent on the position of the surface and its orientation e.g. as front or back surface of a single lens, concerning the sign convention. This means that in this case, a factor of $F_s = -1$ may change the aspheric data of the surface.

So, if a surface is scaled in all dimensions by a scale factor F_s, then the sagitta function is scaled as

$$\widetilde{z(h)} = F_s\, z(h) \tag{40}$$

Comparing the different types of aspheric descriptions listed above, the following changes of the surface data have to be made if the surface is just simply scaled by keeping the ‘asphericity’:

Radius of curvature: $\tilde{R} = F_s R$ (41)

Surface height: $\tilde{h} = \sqrt{(F_s x)^2 + (F_s y)^2} \rightarrow \tilde{h} = |F_s| h,$

$$h_{\max} = |F_s| h_{\max}, \quad \tilde{u} = u \tag{42}$$

Conic constant: $\tilde{\kappa} = \kappa$ (43)

Coefficients of power series
according to (3): $\widetilde{A_{2n}} = F_s^{-2n+1} A_{2n}$ (44)

Coefficients of normalized power series
according to (5): $\widetilde{B_{2n}} = B_{2n}$ (45)

Coefficients of Zernike polynomials
according to (12): $\widetilde{\bar{C}_{2n}} = F_s \bar{C}_{2n}$ (46)

Coefficients of normalized Zernike polynomials
according to (11): $\widetilde{C_{2n}} = C_{2n}$ (47)

Coefficients of Q^{con}-polynomials
according to (16): $\widetilde{a_m} = F_s a_m$ (48)

Coefficients of Q^{*}-polynomials
according to (24): $\widetilde{b_m} = F_s b_m$ (49)

It has to be pointed out here that describing an aspherical surface using a normalized power series or normalized Zernike polynomials, the corresponding expansion coefficients remain constant while scaling [see (45) and (47)], even especially while changing the orientation of the surface. This means that in the optical design process, the data of the expansion functions are decoupled from the basic surface function. Using one of these kinds of description forms might be of great advantage for the optimization of optical systems, and this may also avoid errors in the preparation of technical drawings corresponding to [1, 13].

6 Aspheric description functions for translationally invariant surfaces

So far, rotationally invariant surfaces have been discussed. In a next step, the description forms for such surfaces can be applied for translationally invariant surfaces like cylindrical surfaces, also in such a way that the formalism will be reduced to single dimension x or y. So, instead of describing the sagitta function z as a function of the two-dimensional surface height h as defined in (1), the surface will be described in one shape for x or y only. The perpendicular shape is plano with $\rho_x=0$ or $\rho_y=0$. Therefore, the base parameters will be modified in the following way:

Surface height: $h=x \quad \text{or} \quad h=y$ (50a)

$$h_{\max}=x_{\max} \quad \text{or} \quad h_{\max}=y_{\max} \tag{50b}$$

$$u=u_x=\frac{x}{x_{\max}} \quad \text{or} \quad u=u_y=\frac{y}{y_{\max}} \tag{50c}$$

Curvature: $\rho=\rho_x \quad \text{or} \quad \rho=\rho_y$ (51)

Conic constant: $\kappa=\kappa_x \quad \text{or} \quad \kappa=\kappa_y$ (52)

So, the description of non-circular cylindrical surfaces can be described using similar formulas as for aspheric surfaces, here selected for the x coordinate and summarized as

Standard power series:

$$z(x)=\frac{x^2\rho_x}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2}}+\sum_{n=2}^{N}A_{2n}x^{2n} \tag{53}$$

Normalized power series:

$$z(x)=\frac{x^2\rho_x}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2}}+\frac{1}{\rho}\sum_{n=2}^{N}B_{2n}(x\rho_x)^{2n} \tag{54}$$

Zernike polynomials:

$$z(x)=\frac{x^2\rho_x}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2}}+\frac{1}{\rho_x}\sum_{n=2}^{N}C_{2n}R_{2n}^{4}(u_x) \tag{55}$$

Q^{con}-polynomials:

$$z(x)=\frac{x^2\rho_x}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2}}+u_x^4\sum_{m=0}^{M}a_mQ_m^{\text{con}}(u_x) \tag{56}$$

Q^*-polynomials:

$$z(x)=\frac{x^2\rho_x}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2}}+\frac{u_x^2(1-u_x^2)}{\sqrt{1-(x\rho_x)^2}}\sum_{m=0}^{M}b_mQ_m^*(u_x) \tag{57}$$

As for aspherical surfaces according to (29), the description using Q^*-polynomials (57) can be based on a circular cylindrical form with $\kappa_x=0$ or $\kappa_y=0$, for which the radius of curvature becomes the 'best fit cylindrical radius' as $\rho_{bfc,x}$ or $\rho_{bfc,y}$ fitted to the cylinder vertex line and the edges of the surface.

7 Aspheric description functions for surfaces of less symmetry

More complex surfaces with less symmetry can be developed from the 2-dimensional functions above when applying the formalism for both directions for curved shapes in x and y. If at least one shape is circular, toric surfaces can be described in this way. For the general case of non-circular shapes in both directions and as a special way of describing conoidal surfaces, the aspheric description forms can be applied for 2-dimensional sagitta functions as

$$z(x,y)=\frac{x^2\rho_x+y^2\rho_y}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2-(1+\kappa_y)(y\rho_y)^2}}+\sum_{n=1}^{N}(A_n|x|^n+B_n|y|^n) \tag{58}$$

In principle, also orthogonal polynomials as used for rotationally invariant expansions can be applied. For example, using the Zernike polynomials corresponding to (12) and (13), the sagitta function can be written as

$$z(x,y)=\frac{x^2\rho_x+y^2\rho_y}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2-(1+\kappa_y)(y\rho_y)^2}}+\sum_{n=2}^{N}\left(\frac{1}{\rho_x}C_{2n}R_{2n}^4(u_x)+\frac{1}{\rho_y}D_{2n}R_{2n}^4(u_y)\right) \tag{59a}$$

or

$$z(x,y)=\frac{x^2\rho_x+y^2\rho_y}{1+\sqrt{1-(1+\kappa_x)(x\rho_x)^2-(1+\kappa_y)(y\rho_y)^2}}+\sum_{n=2}^{N}(\bar{C}_{2n}R_{2n}^4(u_x)+\bar{D}_{2n}R_{2n}^4(u_y)) \tag{59b}$$

Even sagitta functions using *Q*-polynomials can be applied in the same way. More complex surfaces up to free-form surfaces are not considered within the ISO standard 10110 Part 12 [1] but can be found in ISO 10110 Part 19 [3] as generalized surfaces. Furthermore, more complex description forms based on (58) but using mixed terms for *x* and *y* can be found in [14–16] and other publications.

8 Summary and conclusion

In this overview, descriptions of aspheric surfaces from types with rotational invariance up to types with less symmetry have be presented, which all are based on conic terms followed by different series expansions. Most of these sagitta functions can be found in the ISO standard 10110 Part 12; other modified versions have been shown here in addition, which may be of special interest. Furthermore, the exact computation of the different polynomials as well as conversion formulas have been listed clearly. The special properties of the different types of description have been discussed. In conclusion, the application of one or another of these types is dependent on the characteristics of optical surfaces within a system and on the usability for the optical design process, the producibility and the measurability as well as for the overall qualification and for the comparability of aspherical surfaces.

References

[1] ISO 10110-12, Optics and photonics – Preparation of drawings for optical elements and systems – Part 12: Aspheric surfaces.
[2] R. Schuhmann, OSA IODC 2017, Proc. of SPIE-OSA 10590, 105900H-1-H7 (2017).
[3] ISO 10110-19, Optics and photonics – Preparation of drawings for optical elements and systems – Part 19: General description of surfaces and components.
[4] R. Schuhmann, Doctoral Thesis (Optical Institute, Technical University Berlin, D83, Berlin, 1985).
[5] J. Kross and R. Schuhmann, Optik 70, 76–85 (1985).
[6] J. Kross, F.-W. Oertmann and R. Schuhmann, Proc. SPIE 656, 300–309 (1986).
[7] R. Schuhmann and G. Adams, Proc. SPIE 4441, 37–42 (2001).
[8] ISO/TR 14999-2, Optics and photonics – Interferometric measurement of optical elements and optical systems – Part 2: Measurement and evaluation techniques.
[9] G. W. Forbes, Opt. Express 15, 5218–5226 (2007).
[10] G. W. Forbes and C. P. Brophy, Opt. Syst. Des. SPIE Proc. 7100, 710002-1-15 (2008).
[11] G. W. Forbes, Opt. Express 18, 19700–19712 (2010).
[12] G. W. Forbes, Opt. Express 18, 13851–13862 (2010).
[13] ISO 10110-1, Optics and photonics – Preparation of drawings for optical elements and systems – Part 1: General.
[14] G. W. Forbes, Opt. Express 20, 2483–2499 (2012).
[15] C. Menke and G. W. Forbes, Adv. Opt. Technol. 2, 97–109 (2012).
[16] A. Yabe, Appl. Opt. 51, 3054–3058 (2012).

Rainer Schuhmann
MSE Metrology and Software Development
Berliner Glas KGaA Herbert Kubatz
GmbH & Co., Waldkraiburger Str. 5
10823 Berlin, Germany
rainer.schuhmann@berlinerglas.de

Rainer Schuhmann is currently Manager of Metrology and Software Department at Berliner Glas in Berlin. After receipt of Diploma and PhD (Dr.-Ing.) in Physics at the Technical University Berlin in the field of Technical Optics and Optical Design here worked with Schneider Kreuznach as Head of Optical Design Department, with LINOS, former Spindler & Hoyer, in Göttingen as Director R&D and Head of Quality Management and finally as Vice President and Head of Business Division Industrial Manufacturing, with Acritec in Hennigsdorf near Berlin as Managing Director in the sector of Ophthalmic Implants (intra-ocular lenses), and he joined Berliner Glas in 2006. Since 1985 he is member of the German Society of Applied Optics (Deutsche Gesellschaft für angewandte Optik (DGaO), the German Branch of the European Optical Society, from 1998 to 2012 board member of DGaO, first as representative to the EOS Advisory Board, from 2004 to 2008 as President of DGaO, in 2001 he directed the 102nd Annual Meeting of the DGaO in Göttingen. Furthermore he is a longtime member of the European Optical Society (EOS) and of the Optical Society of America (OSA), here today as Senior member. Besides activities in various working groups for standardization at DIN (German Institute for Standardization) he was chairman of Subcommittee SC1 'Fundamental Standards' within Technical Committee TC 172 'Optics and Photonics' of the International Organization for Standardization (ISO) from 2009 to 2017.