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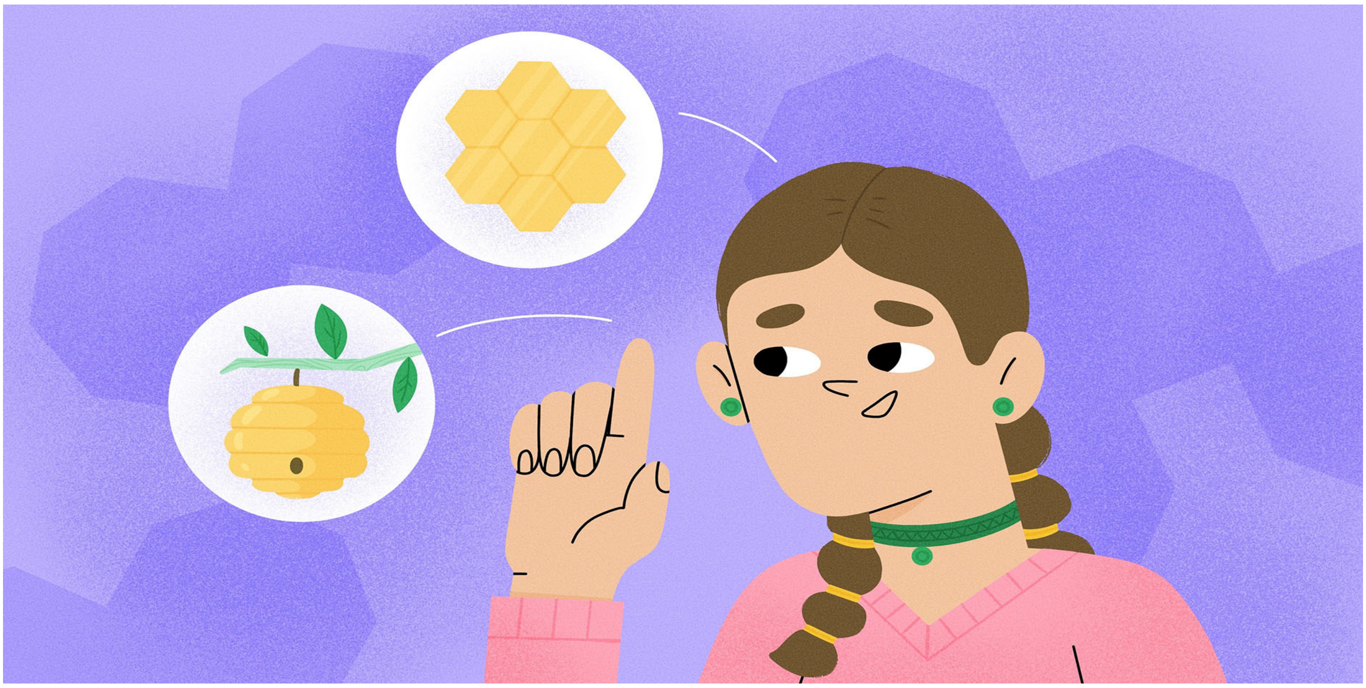

PUZZLING PENTAGONS—HOW MANY WAYS CAN WE COVER A FLAT SURFACE?

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AGE: 11

Convex polygons, shapes bounded by straight lines in which all of the corners point outward, are the simplest of shapes, but despite their simplicity, there are many unsolved problems concerning polygons. Some polygons have the nice property that they fit together espcially well, so that you can use lots of copies of them to cover a large surface to form what is called a tiling. For some polygons, such as triangles, it is easy to see how to form tilings. For others, such as seven-sided convex polygons, it is impossible to form a tiling. This article discusses the history of a long-standing open question in geometry: which convex pentagons give rise to tilings of the plane? The authors also discuss their contribution to the solution of this problem, which involved developing a computerized algorithm to help them search for a new kind of convex pentagon that can form tilings, and the basic idea of this algorithm is discussed.

HOW MATHEMATICAL DISCOVERIES ARE MADE!

Does it surprise you to hear that big, new discoveries in mathematics are being made all the time? Sometimes people think that mathematics mostly involves learning to use formulas discovered in ancient times, such as the Pythagorean theorem. While that is not true, mathematical discoveries can often be difficult to explain in nontechnical terms, and many times these discoveries do not have immediate real-world applications. Indeed, we (the authors) were unaware of new mathematical discoveries until we were college students, so we want to give you an insider's view on how these discoveries can be made.

Like most new science, new mathematical discoveries are born out of curiosity about how the world works. We believe that developing habits of mind that lead to questioning why things are true and whether those truths can apply to new situations can lead to the discovery of new mathematics. People who want to discover new mathematics must be curious about why and how mathematics works, instead of just accepting facts without question! In that spirit, this article tells the story of our own mathematical discovery.

THE TILING PROBLEM: WHAT SHAPES CAN COVER THE PLANE WITHOUT GAPS OR OVERLAPS?

TILING

A tiling of the plane to be an arrangement of shapes that covers the entire infinite 2-dimensional plane without gaps or overlaps.

Tilings are patterns formed from shapes that cover various kinds of surfaces. You have probably seen tiled kitchen floors and patios, or maybe more elaborate examples such as M.C. Escher's artwork. Look around in your daily life, and you will likely see many tilings, as in Figures 1A–C. Tilings appear in the natural world, such as in a bee's honeycomb, the cracked mud of a dried lake bottom, and a giraffe's fur. They are even used to understand how the atoms of crystals fit together. Mathematicians have worked for many years to understand and classify various kinds of tilings.

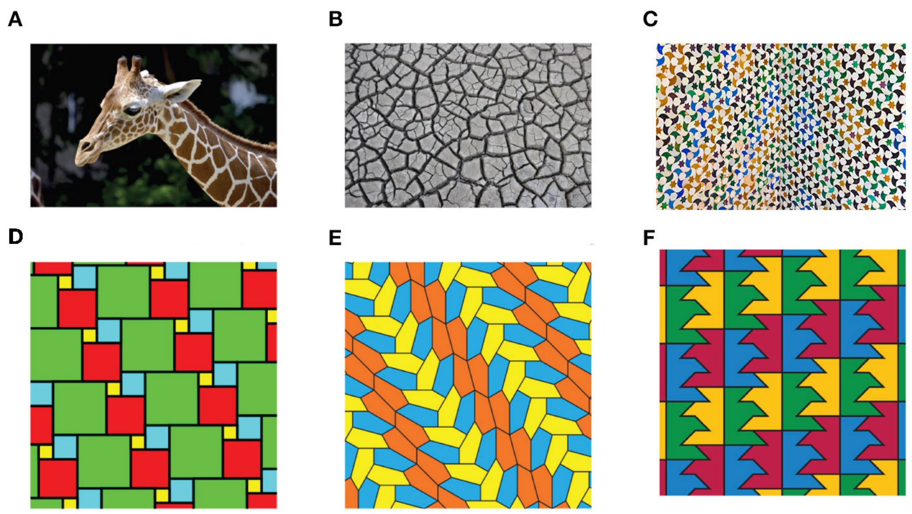

Figure 1

Figure 1

Real-world tilings **(A–C)** and tilings of the plane **(D, E)**. **(A)** Giraffe with tiling fur pattern. **(B)** Dried mud forms a tiling. **(C)** A mosaic pattern at the Alhambra. **(D)** A tiling by squares of 4 sizes. **(E)** A monohedral tiling. **(F)** A tiling by nonconvex polygons.

TILES

The individual shapes in a tiling are called tiles.

MONOHEDRAL TILINGS

Tilings in which all of the tiles are congruent to one another are called monohedral tilings.

TRIANGLES

3-sided polygons are called triangles.

QUADRILATERALS

4-sided polygons are quadrilaterals.

PENTAGONS

5-sided polygons are pentagons.

HEXAGONS

6-sided polygons are called hexagons.

n-GONS

In general, polygons with n sides are called n-gons.

CONVEX POLYGONS

Shapes whose boundaries are straight line segments and whose corners all point outward are called convex polygons.

In geometry, the definition of a tiling is an arrangement of shapes that covers an entire flat surface that goes on forever in all directions, called a plane, without gaps or overlaps (Figures 1D–F). To understand the science of tilings, mathematicians are interested in the central question: "what shapes can be used to form tilings?" This is called The Tiling Problem. Because the shapes in tilings can be so varied (jagged, curvy, pointy, big, or small), it is impossible to answer this question without simplifying it.

To simplify The Tiling Problem so that we may hope to solve it, we must restrict the conditions of the problem in three ways. First, we require that all the individual **tiles** in the tilings have the same shape and size, meaning they are congruent. These are called **monohedral tilings** (Figures 1E, F). Second, we require that the single tile used to create the tiling be a polygon, which is a shape bounded by straight lines, like a triangle (three sides) or a pentagon (five sides). Last, we required that the polygon be convex, meaning its corners all point outward, as in Figure 1E. 3-sided polygons are called **triangles**; 4-sided polygons are **quadrilaterals**; 5-sided polygons are **pentagons**; 6-sided polygons are called **hexagons**; in general, polygons with n sides are called **n-gons**.

When we use these limitations, our simplified version of The Tiling Problem, called The Tiling Problem for **Convex Polygons**, asks: "Which convex polygons can be used to form monohedral tilings of the plane?" (You can try this for yourself by following the direction in Appendix A). The solution to The Tiling Problem for Convex Polygons has a rich history, spanning over 100 years and involving many people (Table 1). Some of the people who contributed to this problem's

Table 1

History of the tiling problem for convex polygons until 1985.

1918	Karl Reinhardt proved there are exactly three types of convex hexagons that can tile the plane [1].
1918	Karl Reinhardt discovered pentagons of Types 1–5 in Figure 2.
1968	Robert Kershner discovered 3 more types of pentagons (Types 6, 7, and 8) [2].
1972	Iven Niven proves any convex polygon with $n \geq 7$ sides will not tile the plane [3] .
1975	Kershner's work was described in a Scientific American article [4], which was read by two separate people who were intrigued enough by this problem to investigate it for themselves.
1975	Richard James III discovered Type 10.
1977	Marjorie Rice, a homemaker from California discovered Type 9 and 11–13.
1985	Rolf Stein found Type 14.

Table 1

solution were not professional mathematicians; they were just curious people who asked "Why?" We hope this encourages you to do so as well!

ISOHEDRAL TILINGS - RECIPES FOR TILINGS

Imagine that you have two copies of the same tiling, one lying on top of the other so that they are perfectly aligned. Now picture sliding the top copy in some direction so that the top copy again perfectly aligns with the bottom copy. Such a motion is called a **symmetry** of the tiling. Another way to understand the idea of a symmetry of a tiling is to think of a "before picture" and an "after picture"—if you slide a tiling and the before picture and the after picture are exactly the same, then the direction and distance that you slid the tiling is a symmetry of the tiling. To describe the complexity of the symmetries of a monohedral tiling, we can ask whether or not, for any two tiles in that tiling, there is a symmetry of the tiling that moves the first tile to the second tile. If the answer is yes, we say the tiling is **isohedral**.

SYMMETRY

A symmetry of a tiling is a way to move a tiling so that the "before picture" and the "after picture" are identical.

ISOHEDRAL

If a monohedral tiling $\mathcal{T}$ has the property that for any two tiles T_1 and T_2 in $\mathcal{T}$, there is a symmetry of $\mathcal{T}$ that moves T_1 to T_2, we say $\mathcal{T}$ is isohedral.

Isohedral tilings look the same around each tile, so we can understand how the shapes fit together in the whole tiling just by understanding what is happening around the boundary of any one tile. The way tiles fit around each other in an isohedral tiling is described by an incidence symbol, which can be viewed as a sort of recipe for how a shape can tile the plane (see Appendix B to explore the idea of incidence symbols.)

HOW WE FOUND A NEW TYPE OF PENTAGON

Led by our curiosity, we began to look for common features of the types of tilings known at the time (types 1–14 in Figure 2). We made two important observations, which generated even more questions!

First, types 1–5 can produce isohedral tilings. In types 6–14, if you cluster together two or three pentagons to act like a single tile, then that cluster produces an isohedral tiling; some types form tilings by clusters of two pentagons, as in Figures 2G, L, and others form tilings by clusters of three pentagons, as in Figures 2J, N. This observation guided us to consider how we might discover new convex pentagons that similarly tile the plane in isohedral clusters.

Our second observation was, along the boundaries of the clusters in tilings of types 6–14, we saw that the middle part of the edges of some pentagons met the corners of other pentagons, as illustrated by the red points in Figure 2N. We call these points **flat nodes**. We wondered how many flat nodes there might be in each cluster, and we proved that in tilings involving clusters of two pentagons, there can be at most two flat nodes per cluster. Similarly, in tilings involving clusters

FLAT NODE

A flat node in a k-block transitive tiling by pentagons is a point on the boundary of a k-block where the corner of one pentagon meets a point in the middle of an edge of another pentagon.

K-BLOCK TRANSITIVE

k-block transitive tilings are those in which identical clusters of k pentagons, acting as a single tile, form an isohedral tiling.

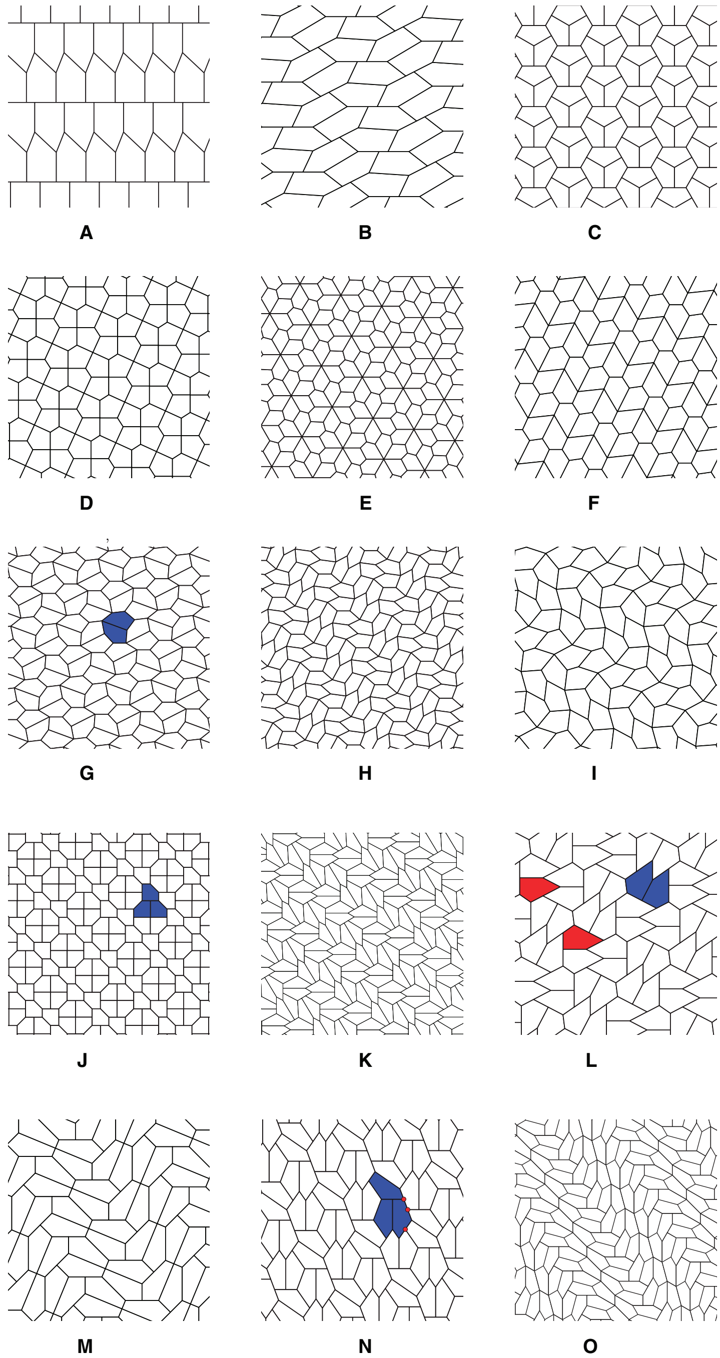

Figure 2

Figure 2

The 15 types of convex pentagons that can tessellate the plane. **(A)** Type 1: $D + E = 180°$. **(B)** Type 2: $C + E = 180°$; $a = d$. **(C)** Type 3: $A = C = D = 120°$; $a = b, d = c + e$. **(D)** Type 4: $A = C = 90°$; $a = b, c = d$. **(E)** Type 5: $C = 2A = 90°$; $a = b$, $c = d$. **(F)** Type 6: $C + E = 180°$, $A = 2C$; $a = b = e, c = d$. **(G)** Type 7: $2B + C = 360°$, $2D + A = 360°$; $a = b = c = d$. **(H)** Type 8: $2A + B = 360°$, $2D + C = 360°$; $a = b = c = d$. **(I)** Type 9: $2E + B = 360°$, $2D + C = 360°$; $a = b = c = d$. **(J)** Type 10: $E = 90°$, $A + D = 180°$, $2B - D = 180°$, $2C + D = 360°$; $a = e = b + d$. **(K)** Type 11: $A = 90°$, $C + E = 180°$, $2B + C = 360°$; $d = e = 2a + c$. **(L)** Type 12: $A = 90°$, $C + E = 180°$, $2B + C = 360°$; $2a = c + e = d$. **(M)** Type 13: $A = C = 90°$, $2B = 2E = 360° - D$; $c = d$, $2c = e$. **(N)** Type 14: $D = 90°$, $2E + A = 360°$, $A + C = 180°$; $b = c = 2a = 2d$. **(O)** Type 15: $A = 60°$, $B = 135°$, $C = 105°$, $D = 90°$, $E = 150°$; $a = 2b = 2d = 2e$.

of three pentagons, there are at most three flat nodes. This was crucial to understanding all the possibilities.

Using these observations, we developed a computer algorithm to search for all possible pentagons that can produce these clustered tilings in which the sizes of the clusters are up to four. Type 15 (Figure 2O) was discovered through this computerized search. The algorithm works as follows. First, flat nodes are placed on the boundary of a template cluster of three pentagons. Second, an isohedral type was chosen for the cluster (out of 81 possible types) and the cluster was labeled according to the “recipe” for that isohedral type. Third, copies of the clusters were placed around a central copy of the cluster, according to the isohedral recipe. Fourth, the corners and sides of the individual pentagons of the clusters were labeled (there are many ways to do this, and the computer checks them all). Finally, from the resulting figure, equations describing the sides and angles of the pentagons were generated. The steps of this process are illustrated in Figure 3.

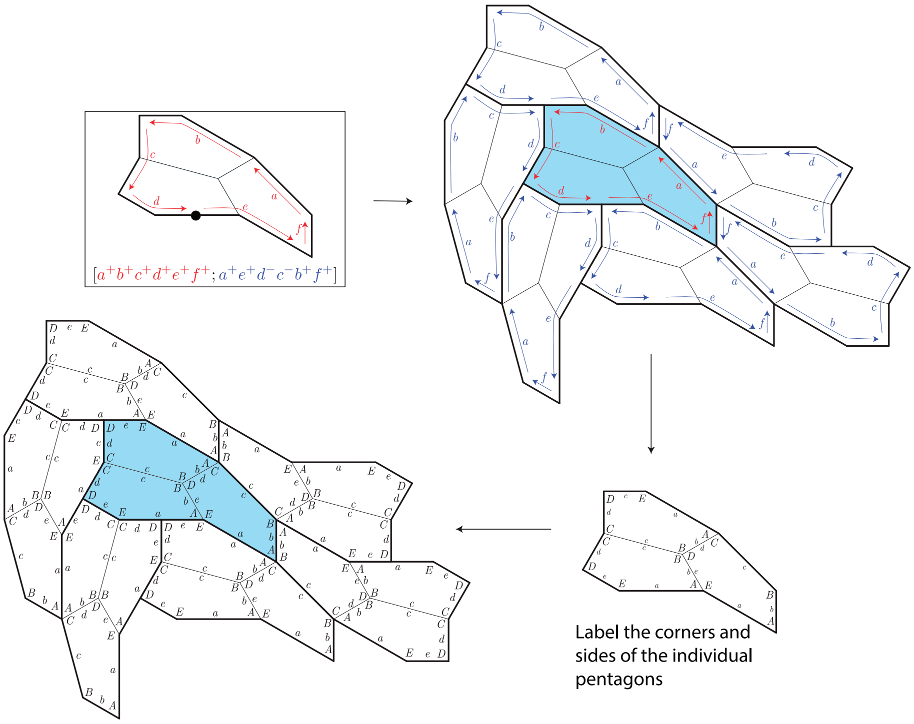

Figure 3

Figure 3

A 3-block cluster that generates an isohedral tiling. The red and blue labels and arrows indicate the incidence symbol $[a^+b^+c^+d^+e^+f^+; a^+e^+d^-c^-b^+f^+]$. To understand the incidence symbol, see Appendix B.

For the choice of labeling of pentagons at right in Figure 3, we see in the center of the figure that there are two corners labeled B and one corner labeled D; that arrangement gives us Equation $2B + D = 360^\circ$. We also see relationships among the sides; toward the middle of the figure, that $b = d$ and $b = e$, and on the right side, taking into account the side labeled a extends over a flat node, we see that $a = e+d$. Writing down all such equations, we get the following equations:

$$\begin{aligned} 2A + B + C &= 360^\circ \\ 2E + A &= 360^\circ \\ 2D + 180^\circ &= 360^\circ \\ 2C + E &= 360^\circ \\ 2B + D &= 360^\circ \end{aligned}$$

$$e = b = d$$
$$a = e + d$$

These equations simplify those in Figure 2O. To justify this was a new type, we verified that we could make a pentagon with the specific angles and side lengths, and that such a pentagon did not satisfy any of the equations that define Types 1–14 (check for yourself!).

After our discovery of Type 15, French mathematician Micheal Rao heard the news and decided to investigate the problem. In doing so, he proved that the pentagon we discovered completed the classification, so types 1–15 form the complete classification of pentagons that tile the plane ([5][1]), settling the longstanding Tiling Problem for Convex Polygons!

[1] M. Rao posted an article on the online repository ArXiv, but the article has not yet been verified through a process of peer review.

SUMMARY

Often, in discovering new mathematical facts and theories, it truly "takes a village" in which individuals contribute in ways big and small to help solve a big problem, and this has certainly been the case with the tiling problem for convex polygons. Another important part of discovery is curiosity. We encourage you to always ask why whenever you encounter a new mathematical fact, formula, or procedure. Try to answer that question for yourself, because that is how we arrive at the new truths of mathematics and science.

REFERENCES

1. Reinhardt, K. 1918. *Über die Zerlegung der Ebene in Polygone*. Frankfurt: Univ. Frankfurt a.M. Noske.
2. Kershner, R. B. 1968. On paving the plane. *Am. Math Mon*. 75:839–44.
3. Niven, I. 1978. Convex polygons that cannot tile the plane. *Am. Math. Mon.* 85:785–92. doi: 10.2307/2320624
4. Gardner, M. 1975. Mathematical games. *Sci. Am.* 233:112–9.
5. Rao, M. 2017. Exhaustive search of convex pentagons which tile the plane. *arXiv*. doi: 10.48550/arXiv.1708.00274

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CONFLICT OF INTEREST: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

YOUNG REVIEWERS

ELLE, AGE: 13

My name is Elle. I like to learn new things to better understand the world around me. I like entering the science fair every year. I like cats, fashion, reading and listening to music.

GIRL SCOUT TROOP 3000, AGE: 12

Troop 3000 started in kindergarten and are now in seventh grade. They enjoy selling and eating cookies and going on outings together. They have collected puzzles and books for nursing homes and for their bronze award, they painted rocks and hid them in various places to bring joy to people.

JOSI, AGE: 11

My name is Josi. I love to build things and explore how things work. Science is one of my favorite subjects. I have won several awards in the science fair.

AUTHORS

CASEY MANN

Casey Mann is an American mathematician specializing in geometry. Professor Mann earned his Ph.D. in Mathematics from the University of Arkansas in 2001. From 2002 to 2013, he worked as a professor at the University of Texas at Tyler, and from 2013 to the present he has been a professor of Mathematics at the University of Washington Bothell. During his time as a professor, Prof. Mann has specialized in research in tiling theory and knot theory. In tiling theory, he is known for his work on Heesch's Tiling problem and for being a co-discoverer of the 15th type of pentagon that tiles the plane. Prof. Mann is also passionate about teaching and mentoring of undergraduate students and he currently serves as the principle investigator of a National Science Foundation funded Research Experiences for Undergraduates site.

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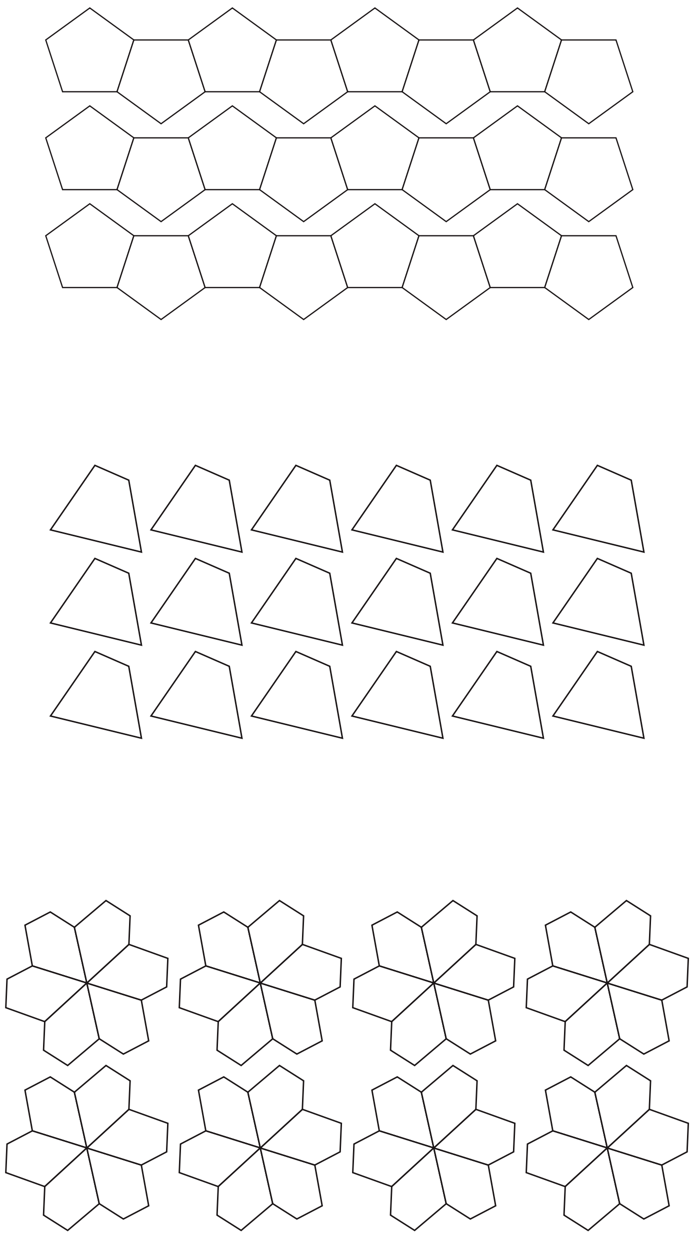
Jennifer McLoud-Mann is an American Indian mathematician whose research interests have included diverse specializations, including commutative algebra, knot theory, and tiling theory. Professor McLoud-Mann worked as a professor at the University of Texas at Tyler from 2002 to 2013 and as a professor at the University of Washington Bothell from 2013 to present, and she has served as Division Chair and as Associate Dean in the School of STEM at the University of Washington Bothell. Prof. McLoud-Mann's research in knot theory has focused on lattice knots and knot mosaics, and her research in tiling theory has focused on pentagonal tilings and on symbolic dynamics associated with Wang tilings. Prof. McLoud is an accomplished teacher and mentor of undergraduate students, receiving the Henry L. Alder Award for Distinguished Teaching in 2009 and serving as principle investigator on two National Science Foundation funded Research Experiences for Undergraduates sites.

APPENDIX A: CUT-OUTS: WHICH OF THESE TILES CAN TESSELLATE THE PLANE?

Directions: Cut out the individual shapes outlined in black and see if you can form tilings using copies of each kind of shape. You might find it easier if you use card stock instead of plain paper. Is a tiling possible or impossible, and why?



APPENDIX B. CUT-OUTS: ISOHEDRAL TYPES

Directions: Cut out the shapes below and assemble them according to the incidence symbol. Lines or arcs of the same color fit together.