frontiers | Frontiers in Physics

TYPE Original Research
PUBLISHED 21 July 2023
DOI 10.3389/fphy.2023.1216451

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RECEIVED 03 May 2023
ACCEPTED 10 July 2023
PUBLISHED 21 July 2023

CITATION
Al-Askar FM and Mohammed WW (2023), Abundant optical solutions for the Sasa-Satsuma equation with M-truncated derivative.
Front. Phys. 11:1216451.
doi: 10.3389/fphy.2023.1216451

Abundant optical solutions for the Sasa-Satsuma equation with M-truncated derivative

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Here, we look at the Sasa-Satsuma equation with M-truncated derivative (SSE-MTD). The analytical solutions in the form of trigonometric, hyperbolic, elliptic, and rational functions are constructed using the Jacobi elliptic function and generalizing Riccati equation mapping methods. Because the Sasa–Satsuma equation is applied to explain the propagation of femtosecond pulses in optical fibers, the acquired solutions can be employed to explain a wide range of important physical phenomena. Moreover, we apply the MATLAB tool to generate a series of graphs to address the effect of the M-truncated derivative on the exact solution of the SSE-MTD.

KEYWORDS

Sasa-Satsuma equation, M-truncated derivative, optical solitons, generalizing Riccati equation mapping method, analytical solutions

1 Introduction

Many authors have centered their attention on fractional nonlinear differential equations (FNLDEs) in the noble age of technology and science to examine complex mathematical models that are used in research area and real life, such as neuroscience, robotics, fluid dynamics, quantum mechanics, plasma physics, optical fibers, and so on. A lot studies have been published about some aspects of fractional differential equations, such as finding exact and numerical solutions, the existence and uniqueness of solutions, and the stability of solutions [1–7]. Therefore, it is essential to discover the exact solutions to these equations in order to understand the physical phenomenon and overcome the resulting obstacles. Recently, acquiring soliton solutions to important equations has emerged as a major field of study. Numerous researchers defended numerous novel methods to evaluate soliton solutions including $(G'/G, 1/G)$-expansion method [8] (G'/G)-expansion [9, 10], generalized (G'/G)-expansion [11], exp-function method [12], Jacobi elliptic function expansion [13], sine-cosine procedure [14], auxiliary equation scheme [15], first-integral method [16], sine-Gordon expansion technique [17], generalized Kudryashov approach [18], $\exp(-\phi(\varsigma))$-expansion method [19], homotopy perturbation method with Aboodh transform [20], He–Laplace method, He's variational iteration method [21, 22], and others.

Abbreviations: 2D, Two dimension; 3D, Three dimension; FNLDEs, Fractional nonlinear differential equations; GREM-method, generalizing Riccati equation mapping method; JEF-method, Jacobi elliptic function method; MTD, M-truncated derivative; ODE, Ordinary differential equation; SSE, Sasa-Satsuma equation; SW, Solitary waves.

In contrast, a new differentiation operator has grown up, that includes the concepts of fractional differentiation and fractal derivative. Therefore, various kinds of fractional derivatives were proposed by several mathematicians. The most well-known ones are the ones proposed by Grunwald-Letnikov, He's fractional derivative, Atangana-Baleanu's derivative, Riemann-Liouville, Marchaud, Riesz, Caputo, Hadamard, Kober, and Erdelyi [23–26]. The bulk of fractional derivative types do not follow classic derivative equations like the chain rule, quotient rule, and product rule. Sousa et al. [27] have developed a new derivative known as the M-truncated derivative (MTD), which is a natural extension of the classical derivative. The MTD for u: $[0,\infty) \to \mathbb{R}$ of order $\delta \in (0, 1]$ is indicated as

$$\mathcal{M}_{j,t}^{\delta,\beta} u(t) = \lim_{h\to 0} \frac{u\left(t\mathcal{E}_{j,\beta}\left(ht^{-\delta}\right)\right) - u(t)}{h},$$

where $\mathcal{E}_{j,\beta}(t)$, for $t \in \mathbb{C}$ and $\beta > 0$, is the truncated Mittag-Leffler function and is defined as:

$$\mathcal{E}_{j,\beta}(t) = \sum_{k=0}^{j} \frac{t^k}{\Gamma(\beta k + 1)}.$$

The MTD has the following characteristics for any real integers a and b [27]:

(1) $\mathcal{M}_{j,t}^{\delta,\beta}(au + bv) = a\mathcal{M}_{j,t}^{\delta,\beta}(u) + b\mathcal{M}_{j,t}^{\delta,\beta}(v)$,

(2) $\mathcal{M}_{j,t}^{\delta,\beta}(u \circ v)(t) = u'(v(t))\mathcal{M}_{j,t}^{\delta,\beta} v(t)$,

(3) $\mathcal{M}_{j,t}^{\delta,\beta}(uv) = u\mathcal{M}_{j,t}^{\delta,\beta} v + v\mathcal{M}_{j,t}^{\delta,\beta} u$,

(4) $\mathcal{M}_{j,t}^{\delta,\beta}(u)(t) = \frac{t^{1-\delta}}{\Gamma(\beta+1)} \frac{du}{dt}$,

(5) $\mathcal{M}_{j,t}^{\delta,\beta}(t^{\nu}) = \frac{\nu}{\Gamma(\beta+1)} t^{\nu-\delta}$.

There are many authors have considered some nonlinear partial differential equations with M-truncated derivative such as [28–31] and the references therein. In this article, we examine the Sasa-Satsuma equation with M-truncated derivative (SSE-MTD):

$$i\mathcal{M}_{j,t}^{\delta,\beta}\mathcal{W} + \frac{1}{2}\mathcal{W}_{xx} + i\left[\alpha_1\mathcal{W}_{xxx} + \alpha_2\mathcal{W}\left(|\mathcal{W}|^2\right)_x + \alpha_3|\mathcal{W}|^2\mathcal{W}_x\right] + \alpha_4|\mathcal{W}|^2\mathcal{W} = 0, \tag{1}$$

where $\mathcal{W} = \mathcal{W}(x,t)$ is the optical soliton profile, $i = \sqrt{-1}$. α_k, for $k =$ 1, 2, 3, 4, are real constants. $\mathcal{W}_t$ defines the temporal evolution of optical soliton molecules, $\mathcal{W}_{xx}$ is the group velocity dispersion. $\mathcal{W}_{xxx}$ represents the third-order dispersion, while $\mathcal{W}(|\mathcal{W}|^2)_x$ provides the stimulated Raman scattering, $|\mathcal{W}|^2\mathcal{W}_x$ is the self-steepening and $|\mathcal{W}|^2\mathcal{W}$ is Kerr-law fiber nonlinearity.

If we set $\delta = 1$ and $\beta = 0$, then we have the Sasa-Satsuma (SS) equation (32, 33):

$$i\mathcal{W}_t + \frac{1}{2}\mathcal{W}_{xx} + i\left[\alpha_1\mathcal{W}_{xxx} + \alpha_2|\mathcal{W}|^2\mathcal{W}_x + \alpha_3\mathcal{W}\left(|\mathcal{W}|^2\right)_x\right] + \alpha_4|\mathcal{W}|^2\mathcal{W} = 0. \tag{2}$$

The SS Eq. 2, which was found while studying the integrability of Schrödinger equation, reduced to nonlinear Schrödinger equation when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ as follows:

$$i\mathcal{W}_t + \frac{1}{2}\mathcal{W}_{xx} + \alpha_4|\mathcal{W}|^2\mathcal{W} = 0. \tag{3}$$

In 1991, Sasa and Satsuma [34] created Eq. 2. This equation has additional components that explain third-order dispersion, self-steepening, and self-frequency shift, which are prevalent in many areas of physics, such as ultrashort pulse propagation in optical fibers [35, 36]. Due to the importance of SS Eq. 2, many authors have obtained its exact solutions by using various methods such as new auxiliary equation method [37], extended trial equation and generalized Kudryashov methods [38], inverse scattering transform [39], improved F-expansion methods and improved auxiliary [40], Riemann problem method [41], unified transform method [42], Bäcklund transformation [43], Darboux transformation [44].

Our main objective of this work is to find the exact solutions of the SSE-MTD (1). The solutions in the form of hyperbolic, trigonometric, elliptic, and rational functions are constructed by utilizing the Jacobi elliptic function method (JEF-method) and generalizing Riccati equation mapping method (GREM-method). Because the Sasa–Satsuma equation is applied to clarify the propagation of femtosecond pulses in optical fibers, the solutions obtained can be employed to study a wide range of important physical phenomena. Furthermore, we utilize the MATLAB tool to generate a series of graphs to examine the effect of the M-truncated derivative on the exact solution of the SSE-MTD (1).

The following is how the paper is organized: In the next section, we describe the methods employed in this paper. The wave equation for the SSE-MTD (1) is developed in Section 3. In Section 4, we employ the JEF-method and the GREM-method to get the precise solutions of the SSE-MTD (1). In Section 5, we study the effect of the MTD on the solution of Eq. 1. Finally, the findings of the article are presented.

2 Description of the methods

In this section, we describe the methods employed in this paper.

2.1 GREM-method

It is useful to outline the essential steps of GREM-method mentioned in [45] as follows:

1. We begin by looking at a general kind of PDEs with MTD as follows

$$\mathcal{P}\left(\mathcal{W}, \mathcal{M}_{j,t}^{\delta,\beta}\mathcal{W}, \mathcal{W}_x, \mathcal{W}_{xx}, \ldots\right) = 0. \tag{4}$$

2. We use Eq. 4 to obtain the traveling wave solution

$$\mathcal{W}(t,x) = \mathcal{X}(\eta_\delta), \quad \eta_\delta = \eta_1 x + \frac{\Gamma(\beta+1)\eta_2}{\delta} t^{\delta}. \tag{5}$$

3. Using the next changes

$$\begin{aligned} \mathcal{M}_{j,t}^{\delta,\beta}\mathcal{W} &= \eta_2\mathcal{X}', \\ u_x &= \eta_1\mathcal{X}', \\ \vdots \quad & \vdots \quad \vdots \\ u_{x^n} &= \eta_1^n\mathcal{X}^{(n)}. \end{aligned} \tag{6}$$

4. After then, substituting (6) into (4) to get ordinary differential equation (ODE)

$$\mathcal{P}\left(\eta_2\mathcal{X}', \eta_1\mathcal{X}', \eta_1^n\mathcal{X}^{(n)}\right) = 0. \tag{7}$$

5. Putting the following Riccati-Bernoulli equation

$$\mathcal{X}' = s\mathcal{X}^2 + r\mathcal{X} + p, \tag{8}$$

where s, r, p are constants, into Eq. 8. Then we balance each coefficient of $\mathcal{X}^k$ to zero to get a system of ODE. We solve this system to attain the value of s, r and p. It is straightforward to obtain the non-traveling wave solutions to Eq. 4 by solving Eq. 8 and utilizing Eq. 5.

2.2 JEF-method

While, we summarize here the main steps of the JEF-method described by Fan et al. [46] as follows.

1. We repeat the first four steps from the previous subsection in order to obtain Eq. 7.
2. Assuming the solution of Eq. 7 in this type

$$\mathcal{X}(\eta_\delta) = \sum_{k=0}^{N} a_k [\mathcal{F}(\eta_\delta)]^k, \tag{9}$$

where N is a positive integer that will be determined and $\mathcal{F}(\eta_\delta) = \mathrm{sn}(\mathcal{K}\eta_\delta, m)$ or $\mathcal{F}(\eta_\delta) = cn(\mathcal{K}\eta_\delta, m)$ or $\mathcal{F}(\eta_\delta) = dn(\mathcal{K}\eta_\delta, m)$ for $0 < m < 1$. The Jacobi elliptic functions $\mathrm{sn}(\mathcal{K}\eta_\delta, m)$, $cn(\mathcal{K}\eta_\delta, m)$, $dn(\mathcal{K}\eta_\delta, m)$ are periodic and have features of triangular functions as follows: $\mathrm{sn}^2(\mathcal{K}\eta_\delta, m) + cn^2(\mathcal{K}\eta_\delta, m) = 1$, $dn^2(\mathcal{K}\eta_\delta, m) = 1 - m^2\mathrm{sn}^2(\mathcal{K}\eta_\delta, m)$, $[\mathrm{sn}(\mathcal{K}\eta_\delta, m)]' = cn(\mathcal{K}\eta_\delta, m) dn(\mathcal{K}\eta_\delta, m)$, $[cn(\mathcal{K}\eta_\delta, m)]' = -\mathrm{sn}(\mathcal{K}\eta_\delta, m) dn(\mathcal{K}\eta_\delta, m)$, $[dn(\mathcal{K}\eta_\delta, m)]' = -m^2\mathrm{sn}(\mathcal{K}\eta_\delta, m) cn(\mathcal{K}\eta_\delta, m)$.

If $m \to 1$, then $\mathrm{sn}(\mathcal{K}\eta_\delta, 1) \to \tanh(\mathcal{K}\eta_\delta)$, $cn(\mathcal{K}\eta_\delta, 1) \to \mathrm{sech}(\mathcal{K}\eta_\delta)$ and $dn(\mathcal{K}\eta_\delta, 1) \to \mathrm{sech}(\mathcal{K}\eta_\delta)$.

3. Usually, to determine the parameter N, we balance the highest order linear terms in the resulting equation with the highest order nonlinear terms. To determine the order, we follow these steps: Firstly, we define the degree of $\mathcal{F}$ as $D[\mathcal{F}] = N$. Secondly, we calculated the highest order nonlinear terms and the highest order nonlinear terms as

$$D\left[\frac{d^n\mathcal{F}}{d\eta^n}\right] = N + n,$$

and

$$D\left[\mathcal{F}^p\left(\frac{d^n\mathcal{F}}{d\eta^n}\right)^s\right] = pN + s(N + n).$$

4. After we determine N, we substitute (9) into the ODE (7) in order to attain an equation in powers of $\mathcal{F}$.
5. Equating each coefficients of powers of $\mathcal{F}$ in the resulting equation to zero. This will provide a set of equations containing the a_k ($k = 0, 1, \ldots N$) and $\mathcal{K}$. We solve these equations to attain the values of a_k ($k = 0, 1, \ldots N$) and $\mathcal{K}$ and substitute with these value into Eq. 9.

3 Traveling wave Eq. For SSE-MTD

To derive the wave equation for SSE-MTD (1), we use

$$\mathcal{W}(x,t) = \mathcal{X}(\eta_\delta)e^{i\mu_\delta}, \ \mu_\delta = \mu_1 x + \frac{\mu_2\Gamma(\beta+1)}{\delta}t^\delta \text{ and}$$
$$\eta_\delta = \eta_1 x + \frac{\eta_2\Gamma(\beta+1)}{\delta}t^\delta, \tag{10}$$

where $\mathcal{X}$ is a real function, μ_1, μ_2, η_1, and η_2 are non-zero constants. We note that

$$\begin{aligned}
\mathcal{M}_{j,t}^{\delta,\beta}\mathcal{W} &= [\eta_2\mathcal{X}' + i\mu_2\mathcal{X}]e^{i\mu_\delta},\\
\mathcal{W}_x &= (\eta_1\mathcal{X}' + i\mu_1\mathcal{X})e^{i\mu_\delta},\ (|\mathcal{W}|^2)_x = \eta_1(\mathcal{X}^2)'e^{i\mu_\delta},\\
\mathcal{W}_{xx} &= \left(\eta_1^2\mathcal{X}'' + 2i\mu_1\eta_1\mathcal{X}' - \mu_1^2\mathcal{X}\right)e^{i\mu_\delta},\\
\mathcal{W}_{xxx} &= \left(\eta_1^3\mathcal{X}''' + 3i\mu_1\eta_1^2\mathcal{X}'' - 3\eta_1\mu_1^2\mathcal{X}' - i\mu_1^3\mathcal{X}\right)e^{i\mu_\delta}.
\end{aligned} \tag{11}$$

Inserting Eq. 11 into Eq. 1, we have for real part

$$\left(\frac{1}{2}\eta_1^2 - 3\alpha_1\mu_1\eta_1^2\right)\mathcal{X}'' + \left(-\mu_2 - \frac{1}{2}\mu_1^2 + \alpha_1\mu_1^3\right)\mathcal{X} + [\alpha_4 - \alpha_1\mu_1]\mathcal{X}^3 = 0, \tag{12}$$

and for imaginary part

$$\alpha_1\eta_1^3\mathcal{X}''' + (\eta_2 + \mu_1\eta_1 - 3\alpha_1\eta_1\mu_1^2)\mathcal{X}' + \eta_1(\alpha_2 + 2\alpha_3)\mathcal{X}^2\mathcal{X}' = 0. \tag{13}$$

Integrating (13) once, we get

$$[\alpha_1\eta_1^3\mathcal{X}'' + (\eta_2 + \mu_1\eta_1 - 3\alpha_1\eta_1\mu_1^2)\mathcal{X} + \frac{1}{3}\eta_1(\alpha_2 + 2\alpha_3)\mathcal{X}^3 = C, \tag{14}$$

where C is the integral constant. If we compare the coefficients of Eqs (12) and (14), we have

$$\begin{aligned}
\eta_1 &= \frac{1}{2\alpha_1} - 3\mu_1,\\
\eta_2 &= -2\gamma_1\mu_1\eta_1 + 3\alpha_1\eta_1\mu_1^2 - \mu_2 - \gamma_1\mu_1^2 + \alpha_1\mu_1^3,\\
\alpha_4 &= \alpha_1\mu_1 + \frac{1}{3}\eta_1(\alpha_2 + 2\alpha_3),
\end{aligned}$$

and

$$C = 0.$$

Now, we can rewrite Eq. 12 as

$$\mathcal{X}'' - \ell_1\mathcal{X}^3 - \ell_2\mathcal{X} = 0, \tag{15}$$

where

$$\ell_1 = \frac{2\alpha_1\mu_1 - 2\alpha_4}{(\eta_1^2 - 6\alpha_1\mu_1\eta_1^2)}, \text{and } \ell_2 = \frac{(2\mu_2 + \mu_1^2 - 2\alpha_1\mu_1^3)}{(\eta_1^2 - 6\alpha_1\mu_1\eta_1^2)}. \tag{16}$$

Balancing $\mathcal{X}''$ with $\mathcal{X}^3$ in Eq. 15 to calculate the parameter N as

$$N + 2 = 3N \Rightarrow N = 1.$$

4 Exact solutions of SSE-MTD

Two various methods such as GREM-method and JEF-method are used to attain the solutions to Eq. 15. The solutions to the SSE-MTD (1) are then determined.

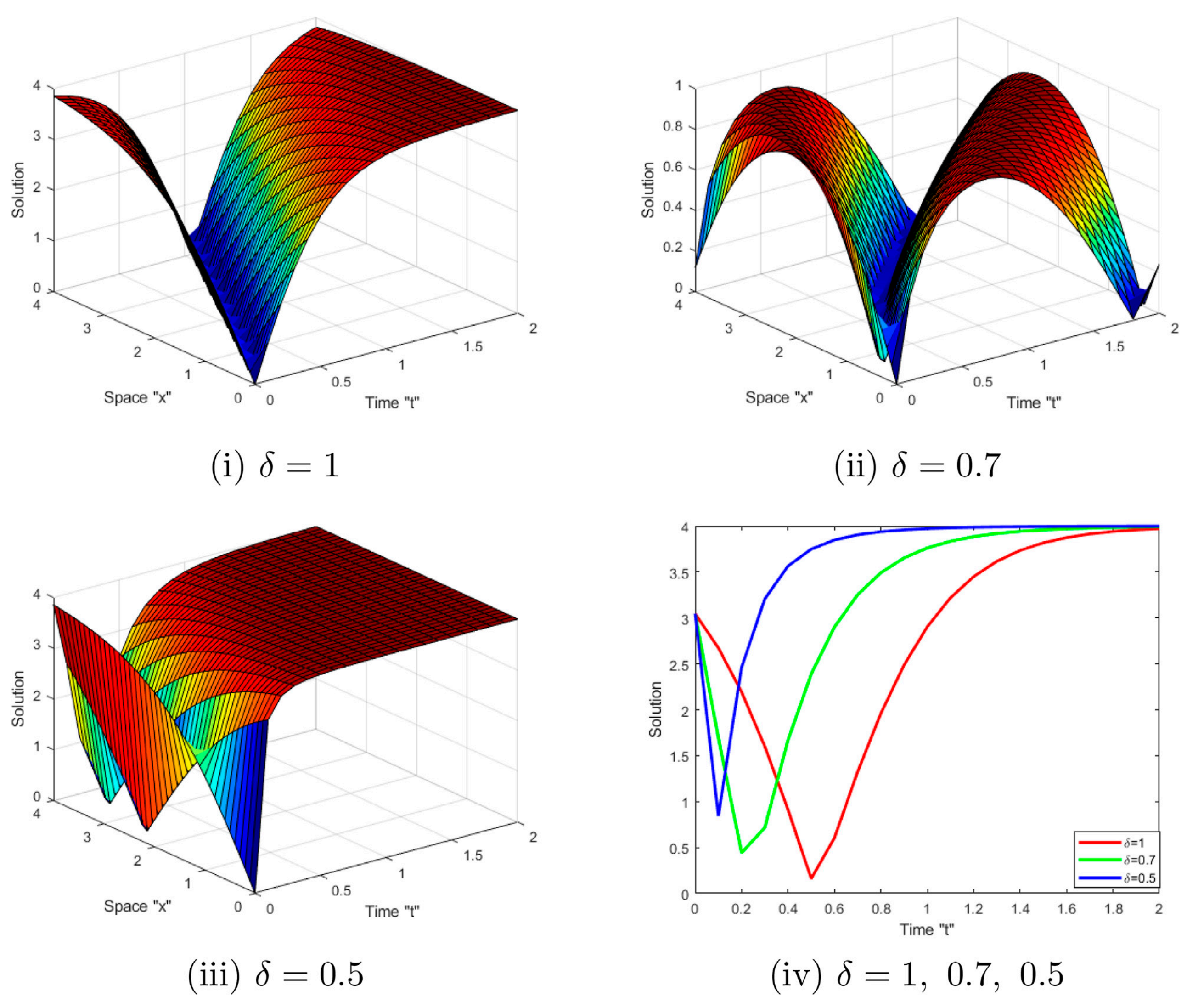

FIGURE 1
(i–iii) display 3D-shape of solution $|\mathcal{W}(x,t)|$ in Eq. 26 with δ =1,0.7,0.5 **(iv)** display 2D-shape of Eq. 26 with different values of δ.

4.1 REM-method

Utilizing Eq. 8, we obtain

$$\mathcal{X}''' = 2s^2\mathcal{X}^3 + 3sr\mathcal{X}^2 + (2sp + r^2)\mathcal{X} + rp. \tag{17}$$

Substituting (17) into (15), we have

$$(2s^2 - \ell_1)\mathcal{X}^3 + 3sr\mathcal{X}^2 + (2sp + r^2 - \ell_2)\mathcal{X} + rp = 0.$$

We put each coefficient of $\mathcal{X}^i$ equal zero in order to get

$$2s^2 - \ell_1 = 0,\ 3sr = 0,\ 2sp + r^2 - \ell_2 = 0, \text{ and } rp = 0.$$

Solving these equations, we have

$$s = \pm\sqrt{\frac{\ell_1}{2}}, \tag{18}$$

$$r = 0, \tag{19}$$

and

$$p = \frac{\ell_2}{2s} = \pm\frac{\ell_2}{\sqrt{2\ell_1}}, \tag{20}$$

where ℓ_1 and ℓ_2 are stated in Eq. 16. There are different sets for the solution of Eq. 8 relying on p and s as follows:

Set I: When $ps > 0$, thus the solutions of Eq. 8 are:

$$\mathcal{X}_1(\eta_\delta) = \sqrt{\frac{p}{s}}\tan\left(\sqrt{ps}\,\eta_\delta\right),$$

$$\mathcal{X}_2(\eta_\delta) = -\sqrt{\frac{p}{s}}\cot\left(\sqrt{ps}\,\eta_\delta\right),$$

$$\mathcal{X}_3(\eta_\delta) = \sqrt{\frac{p}{s}}\left(\tan\left(\sqrt{4ps}\,\eta_\delta\right) \pm \sec\left(\sqrt{4ps}\,\eta_\delta\right)\right),$$

$$\mathcal{X}_4(\eta_\delta) = -\sqrt{\frac{p}{s}}\left(\cot\left(\sqrt{4ps}\,\eta_\delta\right) \pm \csc\left(\sqrt{4ps}\,\eta_\delta\right)\right),$$

$$\mathcal{X}_5(\eta_\delta) = \frac{1}{2}\sqrt{\frac{p}{s}}\left(\tan\left(\frac{1}{2}\sqrt{ps}\,\eta_\delta\right) - \cot\left(\frac{1}{2}\sqrt{ps}\,\eta_\delta\right)\right),$$

Then, SSE-MTD (1) has the trigonometric functions solution:

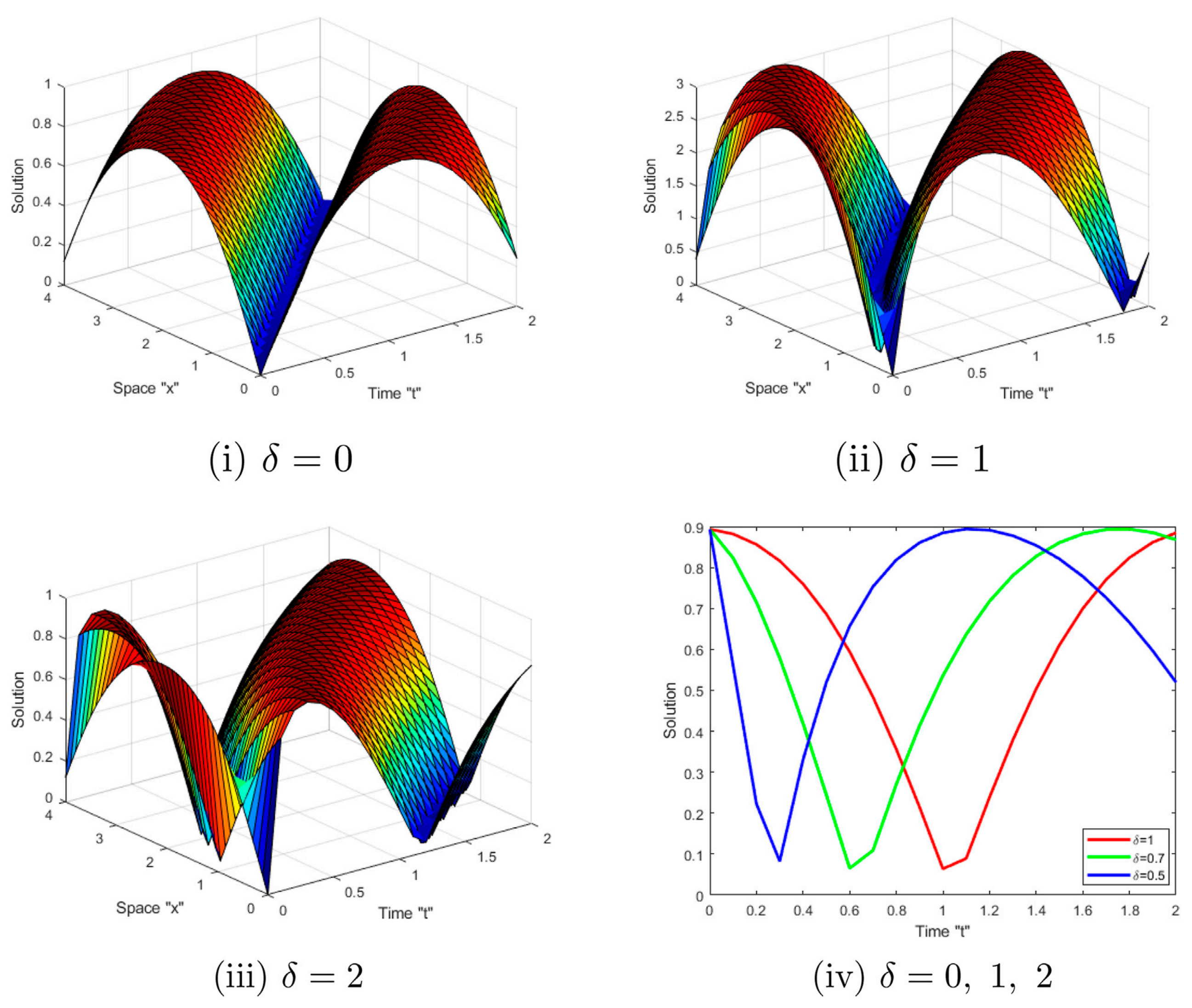

FIGURE 2
(i–iii) display 3D-shape of solution $|\mathcal{W}(x,t)|$ in Eq. 34 with δ = 1,0.7,0.5 **(iv)** display 2D-shape of Eq. 34 with different values of δ.

$$\mathcal{W}_1(x,t) = \sqrt{\frac{p}{s}}\tan\left(\sqrt{ps}\,\eta_\delta\right)e^{i\mu_\delta}, \tag{21}$$

$$\mathcal{W}_2(x,t) = -\sqrt{\frac{p}{s}}\cot\left(\sqrt{ps}\,\eta_\delta\right)e^{i\mu_\delta}, \tag{22}$$

$$\mathcal{W}_3(x,t) = \sqrt{\frac{p}{s}}\left(\tan\left(\sqrt{4ps}\,\eta_\delta\right) \pm \sec\left(\sqrt{4ps}\,\eta_\delta\right)\right)e^{i\mu_\delta}, \tag{23}$$

$$\mathcal{W}_4(x,t) = -\sqrt{\frac{p}{s}}\left(\cot\left(\sqrt{4ps}\,\eta_\delta\right) \pm \csc\left(\sqrt{4ps}\,\eta_\delta\right)\right)e^{\left(\gamma\mathcal{W}(t)-\frac{1}{2}\gamma^2 t\right)}, \tag{24}$$

$$\mathcal{W}_5(x,t) = \frac{1}{2}\sqrt{\frac{p}{s}}\left(\tan\left(\frac{1}{2}\sqrt{ps}\,\eta_\delta\right) - \cot\left(\frac{1}{2}\sqrt{ps}\,\eta_\delta\right)\right)e^{i\mu_\delta}, \tag{25}$$

where $\eta_\delta = \eta_1 x + \frac{\eta_2\Gamma(\beta+1)}{\delta}t^\delta$.

Family II: When $ps < 0$, thus the solutions of Eq. 8 are:

$$\mathcal{X}_6(\eta_\delta) = -\sqrt{\frac{-p}{s}}\tanh\left(\sqrt{-ps}\,\eta_\delta\right),$$

$$\mathcal{X}_7(\eta_\delta) = -\sqrt{\frac{-p}{s}}\coth\left(\sqrt{-ps}\,\eta_\delta\right),$$

$$\mathcal{X}_8(\eta_\delta) = -\sqrt{\frac{-p}{s}}\left(\tanh\left(\sqrt{-4ps}\,\eta_\delta\right) \pm i\operatorname{sech}\left(\sqrt{-4ps}\,\eta_\delta\right)\right),$$

$$\mathcal{X}_9(\eta_\delta) = -\sqrt{\frac{-p}{s}}\left(\coth\left(\sqrt{-4ps}\,\eta_\delta\right) \pm \operatorname{csch}\left(\sqrt{-4ps}\,\eta_\delta\right)\right),$$

$$\mathcal{X}_{10}(\eta_\delta) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\left(\tanh\left(\frac{1}{2}\sqrt{-ps}\,\eta_\delta\right) + \coth\left(\frac{1}{2}\sqrt{-ps}\,\eta_\delta\right)\right).$$

Then, SSE-MTD (1) has the hyperbolic functions solution:

$$\mathcal{W}_6(x,t) = -\sqrt{\frac{-p}{s}}\tanh\left(\sqrt{-ps}\,\eta_\delta\right)e^{i\mu_\delta}, \tag{26}$$

$$\mathcal{W}_7(x,t) = -\sqrt{\frac{-p}{s}}\coth\left(\sqrt{-ps}\,\eta_\delta\right)e^{i\mu_\delta}, \tag{27}$$

$$\mathcal{W}_8(x,t) = -\sqrt{\frac{-p}{s}}\left(\tanh\left(\sqrt{-4ps}\,\eta_\delta\right) \pm i\operatorname{sech}\left(\sqrt{-4ps}\,\eta_\delta\right)\right)e^{i\mu_\delta}, \tag{28}$$

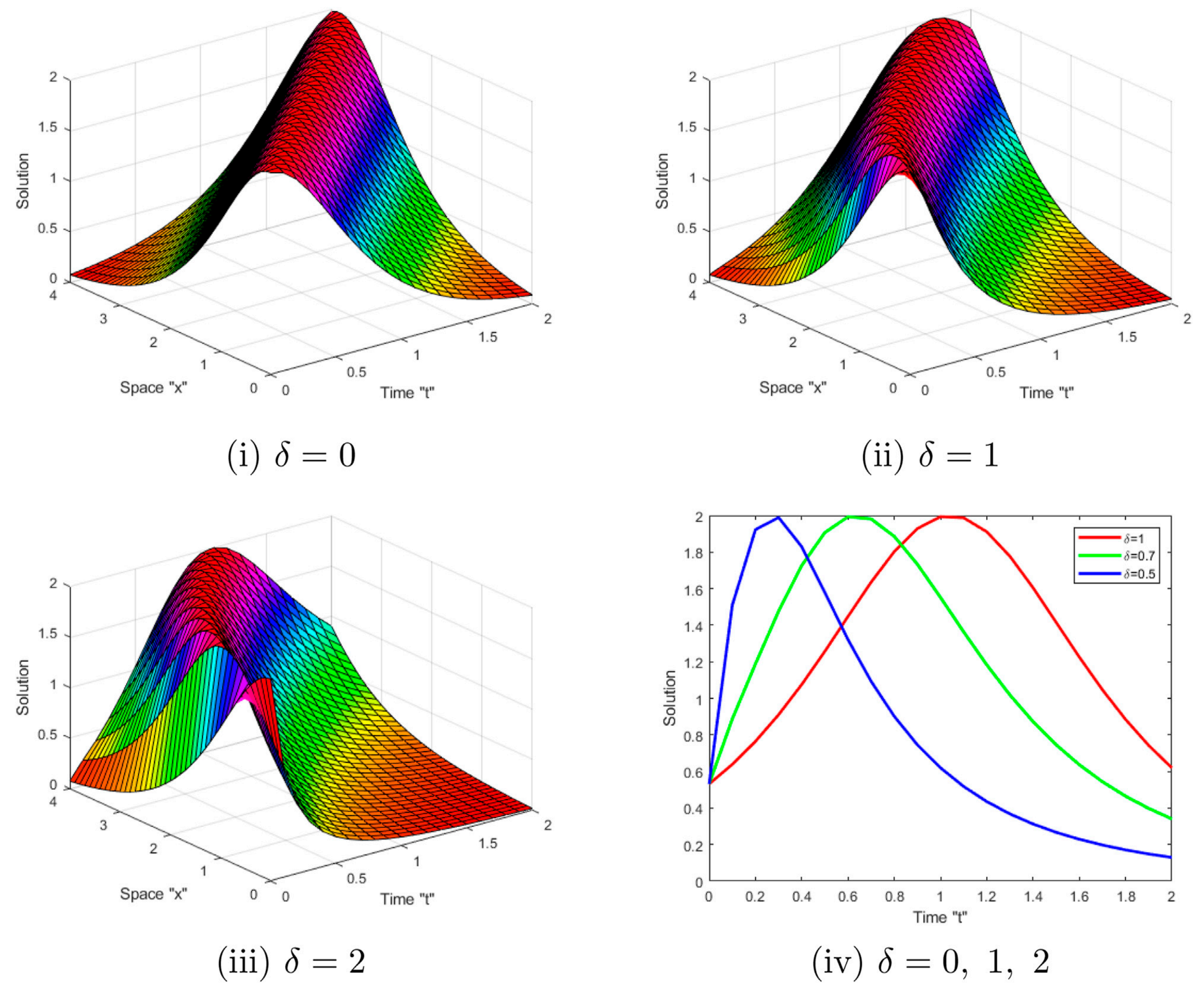

FIGURE 3
(i–iii) display 3D-shape of solution $|\mathcal{W}(x,t)|$ in Eq. 38 with δ = 1, 0.7,0.5 **(iv)** display 2D-shape of Eq. 38 with different values of δ.

$$\mathcal{W}_9(x,t) = -\sqrt{\frac{-p}{s}}\left(\coth\left(\sqrt{-4ps}\,\eta_\delta\right) \pm \operatorname{csch}\left(\sqrt{-4ps}\,\eta_\delta\right)\right)e^{i\mu_\delta}, \tag{29}$$

$$\mathcal{W}_{10}(x,t) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\left(\tanh\left(\frac{1}{2}\sqrt{-ps}\,\eta_\delta\right) + \coth\left(\frac{1}{2}\sqrt{-ps}\,\eta_\delta\right)\right)e^{i\mu_\delta}, \tag{30}$$

where $\eta_\delta = \eta_1 x + \frac{\eta_2\Gamma(\beta+1)}{\delta}t^\delta$.

Family III: When $p = 0$, $s \neq 0$, then the solution of Eq. 8 is

$$\mathcal{X}_{11}(\eta_\delta) = \frac{-1}{s\eta_\delta}.$$

Then, we get the rational function solution of SSE-MTD (1) as

$$\mathcal{W}_{11}(x,t) = \left(\frac{-1}{s\left(\eta_1 x + \frac{\eta_2\Gamma(\beta+1)}{\delta}t^\delta\right)}\right)e^{i\mu_\delta}. \tag{31}$$

Remark 1: *If we Put $\beta = 0$ and $\delta = 0$ in Eqs. (21) and (26), then we get the solutions (13) and (14) that stated in [40].*

4.2 JEF-method

We assume the solutions of Eq. 15, with $N = 1$, are

$$\mathcal{X}(\eta_\delta) = a + b\mathcal{F}(\eta_\delta). \tag{32}$$

First, let $\mathcal{F}(\eta_\delta) = sn(\mathcal{K}\eta_\delta, m)$. Differentiate Eq. 32 two times, we have

$$\mathcal{X}''(\eta_\delta) = -(m^2+1)b\mathcal{K}^2 sn(\mathcal{K}\eta_\delta, m) + 2m^2 b\mathcal{K}^2 sn^3(\mathcal{K}\eta_\delta, m). \tag{33}$$

Setting Eqs 32, 33 into Eq. 15, we obtain

$$(2m^2 b\mathcal{K}^2 - \ell_1 b^3)sn^3(\mathcal{K}\eta_\delta, m) - 3\ell_1 ab^2 sn^2(\mathcal{K}\eta_\delta, m)$$
$$-[(m^2+1)b\mathcal{K}^2 + 3\ell_1 a^2 b + \ell_2 b]sn(\mathcal{K}\eta_\delta, m) - (\ell_1 a^3 + a\ell_2) = 0.$$

Plugging each coefficient of $[\mathrm{sn}(\mathcal{K}\eta_\delta, m)]^n$ equal zero, we attain

$$\ell_1 a^3 + a\ell_2 = 0,$$
$$(m^2+1)b\mathcal{K}^2 + 3\ell_1 a^2 b + \ell_2 b = 0,$$
$$3\ell_1 a b^2 sn^2 = 0,$$

and

$$2m^2 b\mathcal{K}^2 - \ell_1 b^3 = 0.$$

We obtain when we solve these equations

$$a = 0,\ b = \pm\sqrt{\frac{-2m^2\ell_2}{(m^2+1)\ell_1}}\ \mathcal{K}^2 = \frac{-\ell_2}{(m^2+1)}.$$

Consequently, the solution of Eq. 15 is

$$\mathcal{X}(\eta_\delta) = \pm\sqrt{\frac{-2m^2\ell_2}{(m^2+1)\ell_1}}sn\left(\sqrt{\frac{-\ell_2}{(m^2+1)}}\eta_\delta, m\right).$$

As a result, the solution of the SSE-MTD (1), for $\ell_2 < 0$ and $\ell_1 > 0$, is

$$\mathcal{W}(x,t) = \pm\sqrt{\frac{-2m^2\ell_2}{(m^2+1)\ell_1}}sn\left(\sqrt{\frac{-\ell_2}{(m^2+1)}}\eta_\delta, m\right)e^{i\mu_\delta}, \tag{34}$$

where $\eta_\delta = \eta_1 x + \frac{\eta_2\Gamma(\beta+1)}{\delta}t^\delta$. When $m \to 1$, the solution (34) tends to:

$$\mathcal{W}(x,t) = \pm\sqrt{\frac{-\ell_2}{\ell_1}}\tanh\left(\sqrt{\frac{-\ell_2}{2}}(\eta_1 x + \eta_2 t), m\right)e^{i\mu_\delta}. \tag{35}$$

Similarly, we can replace $\mathcal{F}(\eta_\delta)$ in (32) with $cn(\mathcal{K}\eta_\delta, m)$ or $dn(\mathcal{K}\eta_\delta, m)$ to derive the solutions of Eq. 15 as follows:

$$\mathcal{X}(\eta_\delta) = \pm\sqrt{\frac{-2m^2\ell_2}{(2m^2-1)\ell_1}}cn\left(\sqrt{\frac{\ell_2}{(2m^2-1)}}\eta_\delta, m\right),$$

and

$$\mathcal{X}(\eta_\delta) = \pm\sqrt{\frac{-2m^2\ell_2}{(2-m^2)\ell_1}}dn\left(\sqrt{\frac{\ell_2}{(2-m^2)}}\eta_\delta, m\right).$$

Consequently, the solutions of the SSE-MTD (1) are as follows:

$$\mathcal{W}(x,t) = \pm\sqrt{\frac{-2m^2\ell_2}{(2m^2-1)\ell_1}}cn\left(\sqrt{\frac{-\ell_2}{(2m^2-1)}}\eta_\delta, m\right)e^{i\mu_\delta}, \tag{36}$$

for $\frac{\ell_2}{(2m^2-1)} > 0$, $\ell_1 < 0$, and

$$\mathcal{W}(x,t) = \pm\sqrt{\frac{-2m^2\ell_2}{(2-m^2)\ell_1}}dn\left(\sqrt{\frac{\ell_2}{(2-m^2)}}\eta_\delta, m\right)e^{i\mu_\delta}, \tag{37}$$

for $\ell_2 > 0$, $\ell_1 < 0$, respectively. If $m \to 1$, then the solutions (36) and (37) turn to:

$$\mathcal{W}(x,t) = \pm\sqrt{\frac{-2\ell_2}{\ell_1}}\mathrm{sech}\left(\sqrt{\ell_2}(\eta_1 x + \eta_4 t)\right)e^{i\mu_\delta}, \tag{38}$$

for $\ell_2 > 0$, $\ell_1 < 0$.

Remark 2: *If we Put $\beta = 0$ and $\delta = 0$ in Eqs.* (34) *and* (36), *then we get the solutions (48) and (49) that stated in [40].*

5 Discussion and effects of M-truncated derivative

Discussion: For the Sasa-Satsuma equation with a M-truncated derivative, we found the optical solutions in this paper. Two effective methods, the REM-method and JEF-method, were used to arrive at these results. The REM-method has provided optical singular periodic (21) and (22), singular optical solution (27), and dark optical solution (26). While JEF-method has provided elliptic solutions. Dark optical solution can interpret solitary waves (SW) with less intensity than the background [47]. SW with discontinuous derivatives can be illustrated using singular solitons [48, 49]. These kinds of SW are effective because of their efficacy and applicability in optical long-distance communications. Optical fibers can be thought of as thin, long strands of pure-ultra glass that allow electromagnetic radiations to travel unimpeded from one location to another.

Effects of M-truncated derivative: Now, we examine the influence of MTD on the exact solution of the SSE-MTD (1). Several graphical representations depict the behavior of some obtained solutions, including (26) (34) and (38). Let us fix the parameters $\alpha_1 = \frac{1}{2}$, $\mu_1 = \mu_2 = \alpha_4 = \eta_1 = 1$, $\alpha_2 = 2$, $\eta_2 = -2$, $x \in [0,4]$ and $t \in [0, 2]$ to simulate these graphs.

Now, we deduce from Figures 1, 2, 3 that when the derivative order δ of M-truncated derivative increases, the surface moves into the right.

6 Conclusion

In this study, the Sasa-Satsuma equation with M-truncated derivative (SSE-MTD) was examined. We acquired the exact solutions by utilizing Jacobi elliptic function and generalizing Riccati equation mapping methods. Because of the application of the Sasa–Satsuma equation in explaining the propagation of femtosecond pulses in optical fibers, these solutions may explain a wide range of interesting and complex physical phenomena. Furthermore, using the MATLAB program, the M-truncated derivative effects on the exact solutions of SSE-MTD (1) were illustrated. We concluded that when the derivatives order increases the surface moves into the right. In the future work, we can consider Sasa-Satsuma equation with stochastic term.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

WM: software, data curation, formal analysis, investigation, methodology, writing—original draft. FA-A: data curation, investigation, formal analysis, writing—original draft. All authors contributed to the article and approved the submitted version.

Acknowledgments

Princess Nourah bint Abdulrahman University Researcher Supporting Project number (PNURSP2023R 273), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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Nomenclature

$\mathcal{M}_{j,t}^{\delta,\beta}$	M-truncated derivative operator
$\mathcal{E}_{j,\beta}(t)$	Truncated Mittag-Leffler function
$\Gamma(\cdot)$	Gamma function
β	Positive real number
δ	Fractional derivative order
a **and** b	Real constants
α_k, **for** $k = 1, 2, 3, 4$	Real constants
x **and** t	Independent variables
$\mathcal{X}$	Solution of wave equation
η_δ	The wave variable
μ_δ	The phase component
μ_1	The wave frequency
μ_2	The wave number
η_1	The wave frequency
η_2	The wave velocity
Sn	The elliptic sine
Cn	The elliptic cosine
Dn	The delta amplitude
M	Modulus