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Dynamic analysis and optimal control of a stochastic information spreading model considering super-spreader and implicit exposer with random parametric perturbations

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Environmental factors in social systems affect information spreading at all times. This paper proposes a stochastic *S2EIR* model that considers the presence of super-spreaders and implicit exposers in information spreading, as well as the stochastic perturbation of model parameters. The existence of a global positive solution using the *Itô's* formula is then demonstrated. Sufficient conditions for information disappearance and smooth distribution of information are calculated by using the Borel–Cantelli lemma and the strong law of large numbers. Furthermore, the optimal control strategy for the stochastic model is proposed using the Hamiltonian function. The results of the theoretical analysis are supported by numerical simulations and compared to the parameter variations of the deterministic model. The results of this study indicate that white noise facilitates the spread of information. The intensity of perturbation is proportional to the fluctuation of information spreading. Controlling random parameters can effectively facilitate the spread of information. For positive information, the randomness and complexity of the social system should be utilized to increase the spread of information. In contrast, for negative information, randomness in the social system should be suppressed to the greatest extent possible to limit information dissemination.

KEYWORDS

information spreading, stochastic *S2EIR* model, super-spreader and implicit exposer, random parametric perturbations, optimal control

1 Introduction

Information is essential to the development of human society and is the primary means by which humans can comprehend society and improve their cognition. In general, the effects of various types of information on society vary. Information that is beneficial to social development, such as knowledge, innovation, and affinity, must be disseminated [1]; [2]; [3]; [4], while information that is harmful to social development, such as rumors and computer viruses, must be contained [5]; [6]; [7]. Therefore, it is valuable to investigate the characteristics and mechanisms of information dissemination.

Information dissemination is similar to the spread of infectious diseases. Daley and Kendall [8] first applied the classical infectious disease model to the spread of rumors in order to study the spread of harmful information, and they proposed the classical rumor spread *DK* model. The *SI* model [9], the *SIS* model [10], the *SIR* model [11], and other infectious disease models were then used as a research foundation to propose the *SEIR* model [12], the *SPNR* model [13], the *ILSR* model [14], etc.

The aforementioned models typically regard environmental factors as deterministic factors and do not account for them. However, there are many uncertain environmental factors in the real social environment. Random factors, such as government regulation of the media, sudden mass events, and outbreaks of epidemics or geological disasters, can interfere with the dissemination of information [15]. Therefore, researchers are beginning to consider the impact of uncertain environmental factors on the spread of information.

In recent years, extensive studies on stochastic factors in the field of virus propagation have been proposed. Shang [16] developed a low-dimensional system of nonlinear ordinary differential equations to model the mixed susceptible–infected (–recovered) *SI(R)* epidemics on a random network with general degree distributions. Zhang and Li [17] developed the stochastic *SEIR* model with migration and human awareness in complex networks and compared the distance between stochastic and deterministic systems at points of local disease equilibrium. Liu and Jiang [18] developed the stochastic *SEIR* model with asymptomatic infected individuals and calculated the specific form of the probability density near the quasi-endemic equilibrium of the stochastic system. Hussain et al. [19] proposed the stochastic *SIRV* model with general nonlinear incidence and vaccination and argued that stochastic fluctuations could limit the spread of disease. Wang et al. [20] designed the stochastic *SICA* model with standard incidence rates, provided sufficient conditions for HIV extinction and persistence, and discovered that increasing the intensity of stochastic perturbations could control the virus's spread. Bobryk [21] believed that the spread rate in the *SIR* model must be positive, so he incorporated telegraphic noise, trichotomous noise, and bounded noise into the stochastic model and then discussed the effect of the stochastic perturbation on the stability behavior of disease-free equilibrium points. Kiouach and Sabbar [22] used Gaussian white noise and Lévy jumps to represent, respectively, parametric perturbations in stochastic models, and they calculated the disease's mean persistence, ergodicity, and extinction. In addition, the application of infectious disease models with stochastic parameter perturbations to the novel coronavirus has been a hot topic over the past few years [23]; [24]; [25].

In the field of information spread, researchers have focused on the impact of stochastic factors on models of rumor spread. Cheng et al. [26] argued that individual activities play a significant role in rumor spread and that individual activities are affected by many uncertainties. As a result, they constructed the $I_a I_d S_a S_d R_a R_d$ model and analyzed the dynamical behavior of the stochastic model based on this argument. Zhang and Zhu [27] concluded that the spread rate among individuals in rumor spread models is usually disturbed by random factors, and increasing the noise intensity can effectively control the spread of rumors. Tong et al. [28] constructed a

stochastic rumor spread model with media coverage and age structure based on a deterministic model and confirmed the existence of a global positive solution to the model and the necessary conditions for rumor disappearance and smooth distribution. Zhou et al. [29] argued that information intervention influences rumor spread; consequently, they developed a stochastic *SIRZ* rumor spread model considering the information intervention mechanism and proposed an optimal control strategy for rumor control. Yue and Huo [30] developed a stochastic *SICR* rumor spreading model that accounted for media coverage and science education; the study's results demonstrated that media coverage could inhibit rumor spread. Jain et al. [31] proposed the $S_1 S_2 I$ model with expert interaction on a homogeneous network and discovered that noise perturbation caused persistent rumor spread. Huo and Dong [32] developed a stochastic *ISR* model with white noise media coverage, and the results of their study demonstrated that media coverage was inversely proportional to rumor spread. In addition, Mena et al. [33] analyzed the spread characteristics by developing an optimal control model based on the analysis of extinction and persistence of uncertain spread models.

The aforementioned scholars have conducted numerous studies on the influence of random factors on disease spread and rumor spread. In contrast, not all information is harmful to society in the real world, and positive information must be disseminated. Some beneficial information to society may be disseminated by individuals with authority, also known as super-spreaders. Then, it is more likely that the exposers who are in contact with the super-spreaders will comprehend and continue to spread the information. Conversely, the exposers who are not in contact with the super-spreaders are less likely to do so. The former is referred to as explicit spreaders, whereas the latter is referred to as implicit spreaders. This paper develops a stochastic *S2EIR* model that takes super-spreaders and exposers into account. Afterward, the existence of global positive solutions is demonstrated, suitable parameters are selected as control variables after calculating sufficient conditions for information disappearance and smooth information distribution, and the validity of the proposed theorem is validated through numerical simulation. Different from previous studies, this article considers that individuals with strong influence have played a significant role in promoting information spreading while also considering that some individuals are unwilling to spread information even if they are aware of it. Therefore, the common phenomena of "super-spreader" and "asymptomatic infection" in virus spreading are applied to the information spreading model. In addition, for each individual, the contact rate with information spreaders and the spread rate of being an information spreader may be affected by random perturbations in the environment. So, it is more practical to add random factors to the information spreading model. Meanwhile, the optimal control strategy containing random parameters of this paper is based on the optimal values calculated by the control variables.

The remainder of the paper is organized as follows. In Section 2, the stochastic *S2EIR* model taking super-spreaders and exposers into account is constructed. Section 3 proves the existence of a global positive solution. In Section 4, sufficient conditions for the disappearance of information are outlined. Section 5 provides sufficient conditions for stationary information distribution. The existence of optimal control and the optimal control strategy are

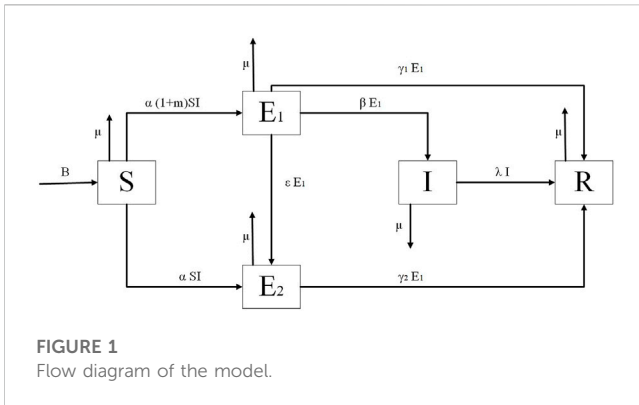


FIGURE 1
Flow diagram of the model.

discussed in Section 6. In Section 7, numerical simulation is used to analyze the impact of stochastic disturbance intensity on information spread and the effect of the control strategy. In the last section, conclusions are provided.

2 The model

This paper examines an open virtual community whose population size fluctuates over time t . The total population size can be represented by $N(t)$, where all populations can be divided into five categories: 1) the receptive population $S(t)$ who do not have access to information but are receptive to it; 2) the explicitly exposed population $E_1(t)$ that has contact with the super-spreaders; 3) the implicitly exposed population $E_2(t)$ that does not have contact with the super-spreaders; 4) the spread-information population $I(t)$; and 5) the non-receptive population $R(t)$ that is no longer interested in the information.

The model flow diagram is presented in Figure 1.

In the model of this paper, the implicit expositors are rather special. People who are not influenced by the information’s super-spreaders typically do not easily comprehend the information’s content since it is spread by super-spreaders. Then, these people demonstrate a lack of interest in the information. Therefore, despite being exposed to the information, these individuals choose not to spread it.

The parameters in the S2EIR model can be interpreted as follows:

- 1) The number of individuals in a social system fluctuates over time. Therefore, in this paper, the author defines B as the number of individuals moving into the social system and defines μ as the emigration rate that is out of the social system due to force majeure factors.
- 2) When information begins to spread through the social system, the receptive population will contact the spreaders with a probability of α . The proportion of super-spreaders is defined as m . Since super-spreaders will speed up the information spread, the spread rate of explicit expositors is defined as $\alpha(1 + m)$, and the spread rate of implicit expositors is represented by α .
- 3) After the expositors make sense of the information, a portion of the explicit expositors who are interested in the information becomes the information spreaders with a probability of β .

- 4) Due to the different degrees of information acceptance among the expositors, the two types of expositors become the non-receptive population with probabilities of γ_1 and γ_2 . Information is generally time-sensitive, and some spreaders lose interest in the information after a period of time. As a result, they become the non-receptive population with a probability of λ .

In addition, the uncertain factors in social systems are commonly referred to as environmental noise. It is not scientific to study the spread of information while ignoring random environmental noise fluctuations. Incorporating environmental noise into deterministic models is more representative of how information spreads in real society. The random factors added to the spread models mainly include three classical approaches: 1) Introducing Gaussian white noise into deterministic parameter perturbation models [34]. 2) Random perturbation encompassing the positive endemic equilibrium of deterministic models [35]. 3) Alternating between regimes based on the probability of Markov chains [36]. Since random perturbations in the environment may affect the contact rate of each individual with the information spreader and the spread rate of becoming an information spreader, this paper uses Gaussian white noise to generate random perturbations of α and β , and the parameters of random perturbation are expressed as follows:

$$\alpha \rightarrow \alpha + \sigma_1 \dot{W}_1(t), \beta \rightarrow \beta + \sigma_2 \dot{W}_2(t). \tag{1}$$

Here, $W_i(i = 1, 2)$ are independent standard Brownian motions and $\sigma_i^2 > 0 (i = 1, 2)$ represent the intensities of $W_i(i = 1, 2)$, respectively. In this paper, W_1 and W_2 represent the relationship without mutual influence between α and β , respectively. In other words, in real scenarios, the conversion rate of information disseminators may be influenced by individual understanding and information acceptability, but it is not affected by the natural contact rate, and there is a state of mutual independence between the two.

The stochastic perturbation parameters are introduced into the deterministic model to construct a stochastic S2EIR model driven by Gaussian white noise, and the stochastic model can be represented as follows:

$$\begin{cases} dS(t) = [B - \alpha(1 + m)SI - \alpha SI - \mu S]dt - \sigma_1(1 + m)SI dW_1(t) - \sigma_1 SI dW_1(t), \\ dE_1(t) = [\alpha(1 + m)SI - \beta E_1 - \gamma_1 E_1 - \epsilon E_1 - \mu E_1]dt + \sigma_1(1 + m)SI dW_1(t) - \sigma_2 E_1 dW_2(t), \\ dE_2(t) = (\alpha SI + \epsilon E_1 - \gamma_2 E_2 - \mu E_2)dt + \sigma_1 SI dW_1(t), \\ dI(t) = (\beta E_1 - \lambda I - \mu I)dt + \sigma_2 E_1 dW_2(t). \end{cases} \tag{2}$$

3 Existence of the global and positive solution

In the rest of this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. In addition, while \mathcal{F}_0 contains all $P - null$ sets, it is

increasing and right continuous [37]. It also can be denoted as follows:

$$\mathbb{R}_+^4 = \{(x_1, x_2, x_3, x_4) | x_i > 0, \quad i = 1, 2, 3, 4\}. \quad (3)$$

The existence of the global solution is the basis of analyzing the dynamic behavior of the stochastic system (Eq. 2). At the same time, according to the actual situation, a positive value is required for the dynamic model of information transmission. The stochastic system (Eq. 2) can be proved as global and positive by Theorem 3.1.

Theorem 3.1. *The existence of a unique positive solution $(S(t), E_1(t), E_2(t), I(t)) \in \mathbb{R}_+^4$ of the stochastic system (Eq. 2) is satisfied by any given initial value $(S(t), E_1(t), E_2(t), I(t)) \in \mathbb{R}_+^4$. The probability of the solution is 1 and remains in \mathbb{R}_+^4 .*

Proof. The existence of a unique local positive solution $(S(t), E_1(t), E_2(t), I(t)) \in \mathbb{R}_+^4$ of the stochastic system (Eq. 2) on $t \in [0, \tau_e)$, which is based on the coefficients of the deterministic system, is locally Lipschitz continuous of any given initial value $(S(0), E_1(0), E_2(0), I(0)) \in \mathbb{R}_+^4$. τ_e is the explosion time [38]. It is required to have $\tau_e = \infty$ a.s. to show this solution globally. The stopping time τ^+ can be defined by

$$\tau^+ = \inf\{t \in [0, \tau_e) : S(t) \geq 0 \text{ or } E_1(t) \geq 0 \text{ or } E_2(t) \geq 0 \text{ or } I(t) \geq 0\}. \quad (4)$$

Let set $\inf \emptyset = \infty$ (\emptyset denotes the empty set). It is easy to get $\tau^+ \leq \tau_e$. So, if $\tau^+ = \infty$ a.s. is proved, then $\tau_e = \infty$ and $(S(t), E_1(t), E_2(t), I(t)) \in \mathbb{R}_+^4$ a.s. for all $t \geq 0$. We assume that $\tau^+ < \infty$; then, $T > 0$ is existence such that $P(\tau^+ < T) > 0$. We define \mathcal{C}^2 function $V: \mathbb{R}_+^4 \rightarrow \mathbb{R}_+^4$ by $V(X) = InSE_1E_2I$. Using the Itô's formula, we calculate the differential of V along the solution trajectories of the stochastic system (2). For $\omega \in (\tau^+ < T)$ and for all $t \in [0, \tau_e)$, we get

$$\begin{aligned} dV(X(t)) = & \left[\frac{B}{S} - \alpha(1+m)I - \alpha I - \mu - \frac{1}{2}\sigma_1^2(1+m)^2I^2 - \frac{1}{2}\sigma_2^2I^2 \right] dt \\ & + \left[\frac{\alpha(1+m)SI}{E_1} - (\beta + \gamma_1 + \varepsilon + \mu) - \frac{1}{2}\sigma_1^2 \frac{(1+m)^2S^2I^2}{E_1^2} - \frac{1}{2}\sigma_2^2 \right] dt \\ & + \left[\frac{\alpha SI}{E_2} + \frac{\varepsilon E_1}{E_2} - (\gamma_2 + \mu) - \frac{1}{2}\sigma_1^2 \frac{S^2I^2}{E_2^2} \right] dt + \left[\frac{\beta E_1}{I} - (\lambda + \mu) \right. \\ & - \left. \frac{1}{2}\sigma_1^2 \frac{E_1^2}{I^2} \right] dt - \sigma_1(1+m)IdW_1 - \sigma_1IdW_1 + \frac{\sigma_1(1+m)SI}{E_1}dW_1 \\ & - \sigma_2dW_2 + \frac{\sigma_1SI}{E_2}dW_1 + \frac{\sigma_2E_1}{I}dW_2. \end{aligned} \quad (5)$$

Positivity of $X(t)$ implies that

$$\begin{aligned} dV(X(t)) \geq & L(S, E_1, E_2, I)dt - \sigma_1 \left[(1+m)I + I - \frac{(1+m)SI}{E_1} - \frac{SI}{E_2} \right] dW_1 \\ & - \sigma_2 \left(1 - \frac{E_1}{I} \right) dW_2, \end{aligned} \quad (6)$$

where

$$\begin{aligned} L(S, E_1, E_2, I) = & -\mu - (\beta + \gamma_1 + \varepsilon + \mu) - (\gamma_2 + \mu) - (\lambda + \mu) - \frac{1}{2}\sigma_1^2(1+m)^2I^2 \\ & - \frac{1}{2}\sigma_1^2I^2 - \frac{1}{2}\sigma_1^2 \frac{(1+m)^2S^2I^2}{E_2^2} - \frac{1}{2}\sigma_1^2 \frac{S^2I^2}{E_2^2} - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\sigma_2^2 \frac{E_1^2}{I^2}. \end{aligned} \quad (7)$$

So, we have

$$\begin{aligned} V(X(t)) \geq & V(X_0) + \int_0^t L(S(u), E_1(u), E_2(u), I(u))du \\ & - \int_0^t \sigma_1 \left[I(u) - \frac{(1+m)S(u)I(u)}{E_1(u)} - \frac{S(u)I(u)}{E_2(u)} \right] dW_1(u) \\ & - \int_0^t \sigma_2 \left(1 - \frac{E_1(u)}{I(u)} \right) dW_2(u). \end{aligned} \quad (8)$$

Note that some components of $X(\tau^+)$ equal 0. Thereby,

$$\lim_{t \rightarrow \tau^+} V(X(t)) = -\infty. \quad (9)$$

Letting $t \rightarrow \tau^+$ in the system (Eq. 8), we have

$$\begin{aligned} -\infty \geq & V(X_0) + \int_0^{\tau^+} L(S(u), E_1(u), E_2(u), I(u))du \\ & - \int_0^{\tau^+} \sigma_1 \left(I(u) - \frac{(1+m)S(u)I(u)}{E_1(u)} - \frac{S(u)I(u)}{E_2(u)} \right) dW_1(u) \\ & - \int_0^{\tau^+} \sigma_2 \left(1 - \frac{E_1(u)}{I(u)} \right) dW_2(u) > -\infty. \end{aligned} \quad (10)$$

According to Eqs. 8–9, it can be obtained that Eq. 10 is less than or equal to $-\infty$. Meanwhile, for any given initial value, $(S(0), E_1(0), E_2(0), I(0)) \in \mathbb{R}_+^4$ and $S(u), E_1(u), E_2(u), I(u)$ in Eq. 10 belong to a positive invariant set and are bounded. Therefore, $S(u), E_1(u), E_2(u), I(u)$ are greater than 0 and greater than $-\infty$, and then, Eq. 10 is greater than $-\infty$. This result is contradictory. In addition, the result obtained by Eq. 10 rejects the original hypothesis $\tau^+ < \infty$. Thus, $\tau^+ = \infty$.

4 Disappearance of the information

Theorems 4.1 and 4.2 give the condition for the disappearance of the information. The condition is expressed by intensities of noises and parameters of the deterministic system. In the stochastic S2EIR model built in this paper, information disappearance needs to meet the following conditions: 1) all expositors affected by the super-spreaders disappeared, and 2) all spreaders of information disappeared. If any of the two aforesaid conditions is satisfied, information can disappear in the social system.

First, Theorem 4.1 gives the condition for the disappearance of information caused by the disappearance of the expositors affected by the super-spreaders.

Theorem 4.1. *For any given initial value $(S(0), E_1(0), E_2(0), I(0)) \in \mathbb{R}_+^4$, $\limsup_{t \rightarrow \infty} \frac{\ln E_1(t)}{t} \leq G(\sigma_1^2, \sigma_2^2)$ holds a.s. Furthermore, $G(\sigma_1^2, \sigma_2^2) < 0$. Then, $E_1(t)$ tend to 0 exponentially a.s., where $G(\sigma_1^2, \sigma_2^2) = \frac{\alpha^2}{2\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2}\sigma_2^2)$.*

Proof. use the Itô's formula to calculate the differentiation of $E_1(t)$ in the stochastic system (2), and $d \ln E_1(t)$ can be written as

$$\begin{aligned} d \ln E_1(t) = & \left[\frac{\alpha(1+m)SI}{E_1} - (\beta + \gamma_1 + \varepsilon + \mu) - \frac{1}{2}\sigma_1^2 \frac{(1+m)^2S^2I^2}{E_1^2} - \frac{1}{2}\sigma_2^2 \right] dt \\ & + \sigma_1 \frac{(1+m)SI}{E_1} dW_1 - \sigma_2 dW_2. \end{aligned} \quad (11)$$

Thus, $\ln E_1(t)$ can be denoted as

$$\ln E_1(t) = \ln E_1(0) + \int_0^t \left[\frac{\alpha(1+m)S(u)I(u)}{E_1(u)} - (\beta + \gamma_1 + \varepsilon + \mu) \right] du + \int_0^t \sigma_1 \frac{(1+m)S(u)I(u)}{E_1(u)} dW_1(u) - \sigma_2 W_2(t) \tag{12}$$

$$\Phi_1(t) = \int_0^t \sigma_1 \frac{(1+m)S(u)I(u)}{E_1(u)} dW_1(u), \tag{13}$$

Here, $\Phi_1(t)$ is a continuous local martingale. The quadratic variation of $\Phi_1(t)$ can be denoted as

$$\langle \Phi_1(t) \rangle = \sigma_1^2 \int_0^t \frac{(1+m)^2 S^2(u) I^2(u)}{E_1^2(u)} du. \tag{14}$$

By exponential martingale inequality [38], it can be known that

$$P \left\{ \sup_{0 \leq t \leq k} \left[\Phi(t) - \frac{c}{2} \langle \Phi(t) \rangle \right] > \frac{2}{c} \ln k \right\} \leq k^{-\frac{2}{c}}, \tag{15}$$

where $0 < c < 1$, k is a random integer. Using the Borel–Cantelli lemma, it is easy to know that the random integer $k_0(\omega)$ exists such that for $k > k_0$, for almost all $\omega \in \Omega$, $\sup_{0 \leq t \leq k} [\Phi(t) - \frac{c}{2} \langle \Phi(t) \rangle] \leq \frac{2}{c}$. Therefore, for all $t \in [0, k]$, we have

$$\int_0^t \sigma_1 \frac{(1+m)S(u)I(u)}{E_1(u)} dW_1(u) \leq \frac{1}{2} c \sigma_1^2 \int_0^t \frac{(1+m)^2 S^2(u) I^2(u)}{E_1^2(u)} du + \frac{2}{c} \ln k. \tag{16}$$

Then, it can be obtained that

$$\ln E_1(t) \leq \ln E_1(0) + \int_0^t \left[\frac{\alpha(1+m)S(u)I(u)}{E_1(u)} - (\beta + \gamma_1 + \varepsilon + \mu) \right] du + \frac{1}{2} (1-c) \sigma_1^2 \int_0^t \frac{(1+m)^2 S^2(u) I^2(u)}{E_1^2(u)} du - \frac{1}{2} \sigma_2^2 t + \frac{2}{c} \ln k - \sigma_2 W_2(t), \tag{17}$$

noting that

$$\frac{\alpha(1+m)S(u)I(u)}{E_1(u)} - \frac{1}{2} (1-c) \sigma_1^2 \frac{(1+m)^2 S^2(u) I^2(u)}{E_1^2(u)} \leq \frac{\alpha^2}{2(1-c)\sigma_1^2}. \tag{18}$$

Substituting Eq. 18 into Eq. 17, $\ln E_1(t)$ can be written as

$$\ln E_1(t) \leq \ln E_1(0) + \int_0^t \left[\frac{\alpha^2}{2(1-c)\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu) - \frac{1}{2} \sigma_2^2 \right] du + \frac{2}{c} \ln k - \sigma_2 W_2(t) = \ln E_1(0) + \left[\frac{\alpha^2}{2(1-c)\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2} \sigma_2^2) \right] t + \frac{2}{c} \ln k - \sigma_2 W_2(t). \tag{19}$$

Hence, for $k-1 \leq t \leq k$, $\frac{\ln E_1(t)}{t}$ can be obtained as

$$\frac{\ln E_1(t)}{t} \leq \frac{\ln E_1(0)}{t} + \frac{\alpha^2}{2(1-c)\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2} \sigma_2^2) + \frac{2}{c} \ln k - \sigma_2 W_2(t). \tag{20}$$

By the strong law of large numbers to the Brownian motion, let $k \rightarrow \infty$ and then $t \rightarrow \infty$. It can be known that $\limsup_{t \rightarrow \infty} \frac{W_2(t)}{t} = 0$. Therefore,

$$\limsup_{t \rightarrow \infty} \frac{\ln E_1(t)}{t} \leq \frac{\alpha^2}{2(1-c)\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2} \sigma_2^2). \tag{21}$$

Finally, let $c \rightarrow 0$. Then, $\limsup_{t \rightarrow \infty} \frac{\ln E_1(t)}{t}$ can be obtained as

$$\limsup_{t \rightarrow \infty} \frac{\ln E_1(t)}{t} \leq \frac{\alpha^2}{2\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2} \sigma_2^2). \tag{22}$$

Next, Theorem 4.2 gives the condition for the disappearance of information caused by the disappearance of the spreaders.

Theorem 4.2. For any given initial value, $(S(0), E_1(0), E_2(0), I(0)) \in \mathbb{R}_+^4$, $\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq G(\sigma_2^2)$ holds a.s. Furthermore, $G(\sigma_2^2) < 0$. Then, $I(t)$ tend to 0 exponentially a.s., where $G(\sigma_2^2) = \frac{\beta^2}{2\sigma_2^2} - (\lambda + \mu)$.

Proof. use the Itô's formula to calculate the differentiation of $I(t)$ in the stochastic system (Eq. 2), and $d \ln I(t)$ can be written as

$$d \ln I(t) = \left[\frac{\beta E_1}{I} - (\lambda + \mu) - \frac{1}{2} \sigma_2^2 \frac{E_1^2}{I^2} \right] dt + \sigma_2 \frac{E_1}{I} dW_2. \tag{23}$$

Thus, $\ln I(t)$ can be denoted as

$$\ln I(t) = \ln I(0) + \int_0^t \left[\frac{\beta E_1(u)}{I(u)} - (\lambda + \mu) - \frac{1}{2} \sigma_2^2 \frac{E_1^2(u)}{I^2(u)} \right] du + \int_0^t \sigma_2 \frac{E_1(u)}{I(u)} dW_2(u). \tag{24}$$

We denote

$$\Phi_2(t) = \int_0^t \sigma_2 \frac{E_1(u)}{I(u)} dW_2(u), \tag{25}$$

where $\Phi_2(t)$ is a continuous local martingale. The quadratic variation of $\Phi_2(t)$ can be denoted as

$$\langle \Phi_2(t) \rangle = \sigma_2^2 \int_0^t \frac{E_1^2(u)}{I^2(u)} du. \tag{26}$$

Similar to Theorem 4.1, for all $t \in [0, k]$, one can obtain

$$\int_0^t \sigma_2 \frac{E_1(u)}{I(u)} dW_2(u) \leq \frac{1}{2} c \sigma_2^2 \int_0^t \frac{E_1^2(u)}{I^2(u)} du + \frac{2}{c} \ln k. \tag{27}$$

Then, it can be obtained that

$$\ln I(t) \leq \ln I(0) + \int_0^t \left[\frac{\beta E_1(u)}{I(u)} - (\lambda + \mu) - \frac{1}{2} (1-c) \sigma_2^2 \frac{E_1^2(u)}{I^2(u)} \right] du + \frac{2}{c} \ln k. \tag{28}$$

It can be noted that

$$\frac{\beta E_1(u)}{I(u)} - \frac{1}{2} (1-c) \sigma_2^2 \frac{E_1^2(u)}{I^2(u)} \leq \frac{\beta^2}{2(1-c)\sigma_2^2}. \tag{29}$$

Substituting Eq. 29 into Eq. 28, $\ln I(t)$ can be written as

$$\begin{aligned} \ln I(t) &\leq \ln I(0) + \int_0^t \left[\frac{\beta^2}{2(1-c)\sigma_2^2} - (\lambda + \mu) \right] du + \frac{2}{c} \ln k \\ &= \ln I(0) + \left[\frac{\beta^2}{2(1-c)\sigma_2^2} - (\lambda + \mu) \right] t + \frac{2}{c} \ln k. \end{aligned} \tag{30}$$

Hence, for $k - 1 \leq t \leq k$, $\frac{\ln I(t)}{t}$ can be obtained as

$$\frac{\ln I(t)}{t} \leq \frac{\ln I(0)}{t} + \frac{\beta^2}{2(1-c)\sigma_2^2} - (\lambda + \mu) + \frac{2}{c} \cdot \frac{\ln k}{k-1}. \tag{31}$$

By the strong law of large numbers to the Brownian motion, let $k \rightarrow \infty$ and then $t \rightarrow \infty$. It can be known that

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \frac{\beta^2}{2(1-c)\sigma_2^2} - (\lambda + \mu). \tag{32}$$

Finally, let $c \rightarrow 0$. Then, $\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t}$ can be obtained as

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \frac{\beta^2}{2\sigma_2^2} - (\lambda + \mu). \tag{33}$$

Remark 4.1: $G(\sigma_1^2, \sigma_2^2) = \frac{\alpha^2}{2\sigma_1^2} - (\beta + \gamma_1 + \varepsilon + \mu + \frac{1}{2}\sigma_2^2)$ and $G(\sigma_2^2) = \frac{\beta^2}{2\sigma_2^2} - (\lambda + \mu)$ are decreasing in σ_1^2 and σ_2^2 . The information will disappear eventually if σ_1^2 and σ_2^2 are large enough, where $G(\sigma_1^2, \sigma_2^2) < 0$ and $G(\sigma_2^2) < 0$.

5 A sufficient condition for the stationary distribution

Theorem 5.1 gives the unique stationary distribution of the existence of the stochastic system (Eq. 2). This also means stability in a stochastic sense.

Theorem 5.1. If $R_0 = \sqrt{\frac{B\alpha\beta(1+m)}{\mu(\beta+\gamma_1+\varepsilon+\mu)(\lambda+\mu)}} > 1$, the stochastic system (Eq. 2) with initial condition $(S(0), E_1(0), E_2(0), I(0)) \in \mathbb{R}_+^4$ and the following conditions are satisfied:

$$0 < \Gamma < \min(\xi_1 S^2, \xi_2 E_1^2, \xi_3 E_2^2, \xi_4 I^2), \tag{34}$$

where

$$\begin{aligned} \Gamma &= 2\sigma_1^2 S^2 I^2 + \frac{1}{2}\sigma_2^2 E_1^* + \sigma_2^2 E_1^{*2}, \\ \xi_1 &= \mu, \\ \xi_2 &= \gamma_1 + \mu - \sigma_2^2, \\ \xi_3 &= \gamma_2 + \mu, \\ \xi_4 &= \lambda + \mu. \end{aligned} \tag{35}$$

Then, the stationary distribution π exists, and the solution of the stochastic system (Eq. 2) is ergodic.

By the information-existence equilibrium point $E^* = (S^*, E_1^*, E_2^*, I^*)$, where $S^* = \frac{B}{\mu R_0^2}$, $E_1^* = \frac{B(1+m)(R_0^2-1)}{(2+m)(\beta+\gamma_1+\varepsilon+\mu)R_0^2}$, $E_2^* = \frac{B(\beta+\gamma_1+\varepsilon+\mu)(R_0^2-1)+B\varepsilon(1+m)(R_0^2-1)}{(2+m)(\gamma_2+\mu)(\beta+\gamma_1+\varepsilon+\mu)R_0^2}$, and $I^* = \frac{\mu(R_0^2-1)}{\alpha(2+m)}$. Then, it can be obtained that

$$\lim_{t \rightarrow \infty} \frac{1}{t} E \int_0^t \left[\xi_1 (S(u) - S^*)^2 + \xi_2 (E_1(u) - E_1^*)^2 + \xi_3 (E_2(u) - E_2^*)^2 + \xi_4 (I(u) - I^*)^2 \right] du < \Gamma. \tag{36}$$

Proof. define a C^2 function V :

$$\begin{aligned} \Theta(S, E_1, E_2, I) &= \Theta_1(S) + \Theta_2(E_1) + \Theta_3(E_2) + \Theta_4(I) \\ &\quad + \Theta_5(S, E_1, E_2, I), \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Theta_1(E_1) &= E_1 - E_1^* - E_1^* \ln \frac{E_1}{E_1^*}, \\ \Theta_2(E_2) &= E_2 - E_2^* - E_2^* \ln \frac{E_2}{E_2^*}, \\ \Theta_3(I) &= I - I^* - I^* \ln \frac{I}{I^*}, \end{aligned} \tag{38}$$

$$\Theta_4(S, E_1, E_2, I) = \frac{1}{2}(S + E_1 + E_2 + I - S^* - E_1^* - E_2^* - I^*)^2.$$

The differential L operator to Θ_1 can be calculated as

$$\begin{aligned} L\Theta_1 &= [\alpha(1+m)SI - (\beta + \gamma_1 + \varepsilon + \mu)E_1] \frac{\partial \Theta_1}{\partial E_1} + \frac{1}{2}(\sigma_1^2(1+m)^2 S^2 I^2 + \sigma_2^2 E_1^2) \frac{\partial^2 \Theta_1}{\partial E_1^2} \\ &= (E_1 - E_1^*) \left[\frac{\alpha(1+m)SI}{E_1} - (\beta + \gamma_1 + \varepsilon + \mu) \right] + \frac{1}{2}\sigma_1^2(1+m)^2 S^2 I^2 + \frac{1}{2}\sigma_2^2 E_1^2. \end{aligned} \tag{39}$$

According to $E^* = (S^*, E_1^*, E_2^*, I^*)$, it is easy to get that

$$\beta + \gamma_1 + \varepsilon + \mu = \frac{\alpha(1+m)S^*I^*}{E_1^*}, \tag{40}$$

and then, $L\Theta_1$ can be expressed as

$$\begin{aligned} L\Theta_1 &= (E_1 - E_1^*) \left[\alpha(1+m) \left(\frac{SI}{E_1} - \frac{S^*I^*}{E_1^*} \right) \right] + \frac{1}{2}\sigma_1^2(1+m)^2 S^2 I^2 + \frac{1}{2}\sigma_2^2 E_1^* \\ &= (E_1 - E_1^*) \left[-\alpha(1+m) \frac{SI(E_1 - E_1^*)}{E_1 E_1^*} + \alpha(1+m) \frac{(S - S^*)(I - I^*)}{E_1^*} \right] \\ &\quad + \frac{1}{2}\sigma_1^2(1+m)^2 S^2 I^2 + \frac{1}{2}\sigma_2^2 E_1^*, \end{aligned} \tag{41}$$

where $\alpha(1+m) \frac{SI(E_1 - E_1^*)}{E_1 E_1^*} \geq 0$ and $E_1^* > 0$.

By simple calculation, one can get

$$\begin{aligned} L\Theta_1 &\leq \alpha(1+m)(S - S^*)(I - I^*)(E_1 - E_1^*) \\ &\quad + \frac{1}{2}\sigma_1^2(1+m)^2 [(S - S^*) + S^*]^2 [(I - I^*) + I^*]^2 + \frac{1}{2}\sigma_2^2 E_1^*. \end{aligned} \tag{42}$$

Due to $\frac{1}{2}(x + y)^2 \leq x^2 + y^2$, it is easy to obtain that

$$\begin{aligned} L\Theta_1 &\leq \alpha(1+m)(S - S^*)(I - I^*)(E_1 - E_1^*) \\ &\quad + \sigma_1^2(1+m)^2 (S - S^*)^2 (I - I^*)^2 + \sigma_1^2 S^2 I^2 + \frac{1}{2}\sigma_2^2 E_1^*. \end{aligned} \tag{43}$$

Similarly, the differential L operator to Θ_2 can be calculated as

$$\begin{aligned} L\Theta_2 &= [\alpha SI + \varepsilon E_1 - \gamma_2 E_2 - \mu E_2] \frac{\partial \Theta_2}{\partial E_2} + \frac{1}{2}(\sigma_1^2 S^2 I^2) \frac{\partial^2 \Theta_2}{\partial E_2^2} \\ &= (E_2 - E_2^*) \left[\frac{\alpha SI}{E_2} + \frac{\varepsilon E_1}{E_2} - (\gamma_2 + \mu) \right] + \frac{1}{2}\sigma_1^2 S^2 I^2. \end{aligned} \tag{44}$$

According to $E^* = (S^*, E_1^*, E_2^*, I^*)$, it is easy to get that

$$\gamma_2 + \mu = \frac{\alpha S^* I^*}{E_2^*} + \frac{\varepsilon E_1^*}{E_2^*}, \tag{45}$$

and then, $L\Theta_2$ can be expressed as

$$\begin{aligned} L\Theta_2 &= (E_2 - E_2^*) \left[\frac{\alpha SI}{E_2} - \frac{\alpha S^* I^*}{E_2^*} + \frac{\varepsilon E_1}{E_2} - \frac{\varepsilon E_1^*}{E_2^*} \right] + \frac{1}{2} \sigma_1^2 S^2 I^2 \\ &= (E_2 - E_2^*) \left[\begin{aligned} &-\alpha \frac{SI(E_2 - E_2^*)}{E_2 E_2^*} - \varepsilon \frac{E_1(E_2 - E_2^*)}{E_2 E_2^*} \\ &+ \alpha \frac{(S - S^*)(I - I^*)}{E_2^*} + \varepsilon \frac{(E_1 - E_1^*)}{E_2^*} \end{aligned} \right] + \frac{1}{2} \sigma_1^2 S^2 I^2, \end{aligned} \tag{46}$$

where $\frac{\alpha SI(E_2 - E_2^*)}{E_2 E_2^*} \geq 0$, $\frac{\varepsilon E_1(E_2 - E_2^*)}{E_2 E_2^*} \geq 0$, and $E_2^* > 0$.

By simple calculation, one can get

$$\begin{aligned} L\Theta_2 &\leq \alpha(S - S^*)(I - I^*)(E_2 - E_2^*) + \varepsilon(E_1 - E_1^*)(E_2 - E_2^*) \\ &\quad + \frac{1}{2} \sigma_1^2 [(S - S^*) + S^*]^2 [(I - I^*) + I^*]^2. \end{aligned} \tag{47}$$

Due to $\frac{1}{2}(x + y)^2 \leq x^2 + y^2$, it is easy to obtain that

$$\begin{aligned} L\Theta_2 &\leq \alpha(S - S^*)(I - I^*)(E_2 - E_2^*) + \varepsilon(E_1 - E_1^*)(E_2 - E_2^*) \\ &\quad + \sigma_1^2 (S - S^*)(I - I^*) + \sigma_1^2 S^2 I^2. \end{aligned} \tag{48}$$

Next, the differential L operator to Θ_3 can be calculated as

$$\begin{aligned} L\Theta_3 &= [\beta E_1 - (\lambda + \mu)I] \frac{\partial \Theta_3}{\partial I} + \frac{1}{2} (\sigma_2^2 E_1^2) \frac{\partial^2 \Theta_3}{\partial I^2} \\ &= (I - I^*) \left[\frac{\beta E_1}{I} - (\lambda + \mu) \right] + \frac{1}{2} \sigma_2^2 E_1^2. \end{aligned} \tag{49}$$

According to $E^* = (S^*, E_1^*, E_2^*, I^*)$, it is easy to get that

$$\lambda + \mu = \frac{\beta E_1^*}{I^*}, \tag{50}$$

and $L\Theta_3$ can be obtained as

$$\begin{aligned} L\Theta_3 &= (I - I^*) \left[\frac{\beta E_1}{I} - \frac{\beta E_1^*}{I^*} \right] + \frac{1}{2} \sigma_2^2 E_1^2 \\ &= (I - I^*) \left[-\frac{\beta E_1(I - I^*)}{II^*} + \frac{\beta(E_1 - E_1^*)}{I^*} \right] + \frac{1}{2} \sigma_2^2 E_1^2, \end{aligned} \tag{51}$$

where $\frac{\beta E_1(I - I^*)}{II^*} \geq 0$ and $I^* > 0$.

By simple calculation, one can get

$$L\Theta_3 \leq \beta(E_1 - E_1^*)(I - I^*) + \frac{1}{2} \sigma_2^2 (E_1 - E_1^* + E_1^*)^2, \tag{52}$$

Due to $\frac{1}{2}(x + y)^2 \leq x^2 + y^2$, it is easy to obtain that

$$L\Theta_3 \leq \beta(E_1 - E_1^*)(I - I^*) + \sigma_2^2 (E_1 - E_1^*)^2 + \sigma_2^2 E_1^2. \tag{53}$$

Finally, the differential L operator to Θ_4 can be calculated as

$$\begin{aligned} L\Theta_4 &= \left(\begin{aligned} &S + E_1 + E_2 + I \\ &-S^* - E_1^* - E_2^* - I^* \end{aligned} \right) [B - \mu S - (\gamma_1 + \mu)E_1 - (\gamma_2 + \mu)E_2 - (\lambda + \mu)I] \\ &= \left(\begin{aligned} &S - S^* + E_1 - E_1^* \\ &+ E_2 - E_2^* + I - I^* \end{aligned} \right) \left[\begin{aligned} &-\mu(S - S^*) - (\gamma_1 + \mu)(E_1 - E_1^*) \\ &-(\gamma_2 + \mu)(E_2 - E_2^*) - (\lambda + \mu)(I - I^*) \end{aligned} \right] \\ &\leq -\mu(S - S^*)^2 - \mu(S - S^*)(E_1 - E_1^*) - \mu(S - S^*)(E_2 - E_2^*) - \mu(S - S^*)(I - I^*) \\ &\quad - (\gamma_1 + \mu)(E_1 - E_1^*)(S - S^*) - (\gamma_1 + \mu)(E_1 - E_1^*)^2 - (\gamma_1 + \mu)(E_1 - E_1^*)(E_2 - E_2^*) \\ &\quad - (\gamma_1 + \mu)(E_1 - E_1^*)(I - I^*) - (\gamma_2 + \mu)(E_2 - E_2^*)(S - S^*) - (\gamma_2 + \mu)(E_2 - E_2^*)^2 \\ &\quad - (\gamma_2 + \mu)(E_2 - E_2^*)(E_1 - E_1^*) - (\gamma_2 + \mu)(E_2 - E_2^*)(I - I^*) - (\lambda + \mu)(I - I^*)^2 \\ &\quad - (\lambda + \mu)(I - I^*)(S - S^*) - (\lambda + \mu)(I - I^*)(E_1 - E_1^*) - (\lambda + \mu)(I - I^*)(E_2 - E_2^*). \end{aligned} \tag{54}$$

Substituting Eqs 43, 48, 53, and 54 into Eq. 37, we get

$$\begin{aligned} \Theta(S, E_1, E_2, I) &\leq \left(\sigma_1^2 S^2 I^2 + \frac{1}{2} \sigma_2^2 E_1^2 \right) + \sigma_1^2 S^2 I^2 + [\sigma_2^2 (E_1 - E_1^*)^2 + \sigma_2^2 E_1^2] \\ &\quad + [-\mu(S - S^*)^2 - (\gamma_1 + \mu)(E_1 - E_1^*)^2 - (\gamma_2 + \mu)(E_2 - E_2^*)^2 - (\lambda + \mu)(I - I^*)^2] \\ &= -\mu(S - S^*)^2 - (\gamma_1 + \mu)(E_1 - E_1^*)^2 - (\gamma_2 + \mu)(E_2 - E_2^*)^2 \\ &\quad - (\lambda + \mu)(I - I^*)^2 + 2\sigma_1^2 S^2 I^2 + \frac{1}{2} \sigma_2^2 E_1^2 + \sigma_2^2 E_1^2. \end{aligned} \tag{55}$$

By Eq. 34, the ellipsoid

$$-\xi_1(S - S^*)^2 - \xi_2(E_1 - E_1^*)^2 - \xi_3(E_2 - E_2^*)^2 - \xi_4(I - I^*)^2 + \Gamma = 0 \tag{56}$$

lies entirely in \mathbb{R}_+^4 . According to [37], it is easy to know that the stochastic system (Eq. 2) has a stable stationary distribution. □

Remark 5.1: By Theorem 5.1, there exist

$$\begin{aligned} \lim_{(\sigma_1, \sigma_2) \rightarrow 0} \Gamma &= 0, \\ \lim_{(\sigma_1, \sigma_2) \rightarrow 0} \xi_1 &= \mu > 0, \\ \lim_{(\sigma_1, \sigma_2) \rightarrow 0} \xi_2 &= \gamma_1 + \mu > 0, \\ \lim_{(\sigma_1, \sigma_2) \rightarrow 0} \xi_3 &= \gamma_2 + \mu > 0, \\ \lim_{(\sigma_1, \sigma_2) \rightarrow 0} \xi_4 &= \lambda + \mu > 0, \end{aligned} \tag{57}$$

so that the solution of the stochastic system (Eq. 2) fluctuates around E^* . Moreover, the difference between the deterministic system and stochastic system (Eq. 2) decreases with the decrease in the values of σ_1, σ_2 .

6 The stochastic optimal control model

Based on the stochastic model described earlier, this paper proposes a control objective to promote the mass spread of information, considering that information, such as science, technology, and knowledge, plays a positive role in social development. Therefore, the two proportional constants α and β in the model are changed into control variables $\alpha(t)$ and $\beta(t)$, respectively, which control the contact rate of the receptive and spread-information population and the proportion of the explicit expositors converted into the spread-information population.

Hence, the objective function can be proposed as

$$J(E_1, I) = \int_0^{t_f} \left[E_1(t) + I(t) - \frac{c_1}{2} \alpha^2(t) - \frac{c_2}{2} \beta^2(t) \right] dt, \tag{58}$$

and the objective function satisfies the state system as

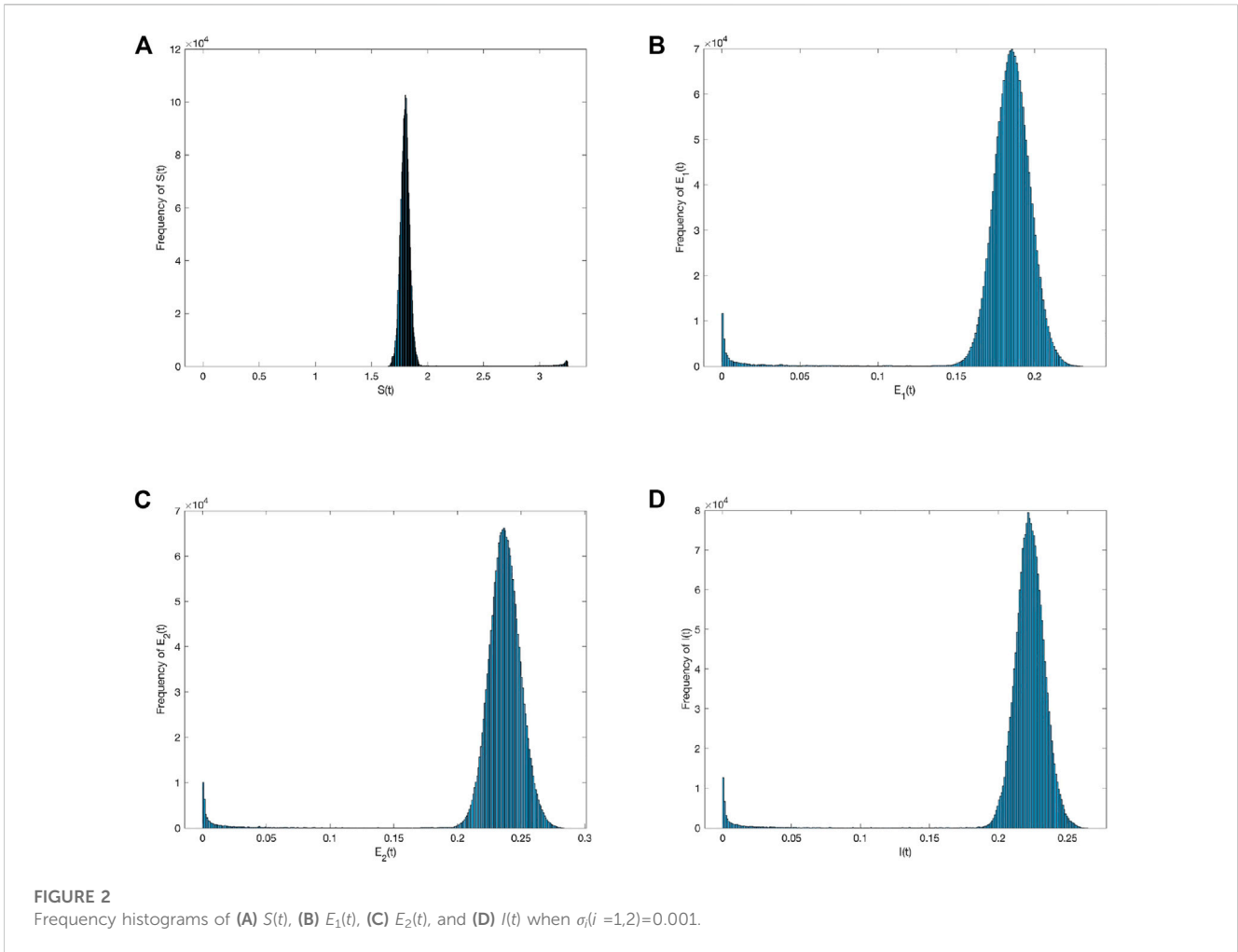
$$\begin{cases} dS(t) = [B - \alpha(t)(1 + m)SI - \alpha(t)SI - \mu S] dt \\ \quad - \sigma_1(1 + m)SI dW_1(t) - \sigma_1 SI dW_1(t), \\ dE_1(t) = [\alpha(t)(1 + m)SI - \beta(t)E_1 - \gamma_1 E_1 - \varepsilon E_1 - \mu E_1] dt \\ \quad + \sigma_1(1 + m)SI dW_1(t) - \sigma_2 E_1 dW_2(t), \\ dE_2(t) = [\alpha(t)SI + \varepsilon E_1 - \gamma_2 E_2 - \mu E_2] dt + \sigma_1 SI dW_1(t), \\ dI(t) = [\beta(t)E_1 - \lambda I - \mu I] dt + \sigma_2 E_1 dW_2(t). \end{cases} \tag{59}$$

The initial conditions for system (Eq. 59) are satisfied:

$$S(0) = S_0, E_1(0) = E_{1,0}, E_2(0) = E_{2,0}, I(0) = I_0, \tag{60}$$

where

$$\alpha(t), \beta(t) \in U_\Delta = \{(\alpha, \beta) \mid (\alpha(t), \beta(t)) \text{ measurable}, 0 \leq \alpha(t), \beta(t) \leq 1, \forall t \in [0, t_f]\}, \tag{61}$$



while U is the admissible control set. 0 and t_f are the time interval. The control strength and importance of control measures are expressed as c_1 and c_2 , respectively, which are the positive weight coefficients.

Theorem 6.1. *There exists an optimal control pair $(\alpha^*, \beta^*) \in U$, so the function is established as*

$$J(\alpha^*, \beta^*) = \max\{J(\alpha, \beta) : (\alpha, \beta) \in U\}. \quad (62)$$

Proof. Let $X(t) = (S(t), E_1(t), E_2(t), I(t))^T$ and

$$L(t; X(t), \alpha(t); \beta(t)) = E_1(t) + I(t) - \frac{c_1}{2}\alpha^2(t) - \frac{c_2}{2}\beta^2(t). \quad (63)$$

The following five conditions must be satisfied, and then, the optimal control pair is in existence.

- i) The set of control variables and state variables is nonempty.
- ii) The control set U is convex and closed.
- iii) The right-hand side of the state system is bounded by a linear function in the state and control variables.
- iv) The integrand of the objective functional is convex on U .

- v) There exist constants $d_1, d_2 > 0$ and $\rho > 1$ such that the integrand of the objective functional satisfied

$$-L(t; X(t), \alpha; \beta) \geq d_1(|\alpha|^2 + |\beta|^2)^\rho - d_2. \quad (64)$$

It is clear that conditions (i)–(iii) are established. Then, condition (iv) can be easily established such that

$$S(t) \leq B, E_1(t) \leq \alpha(t)(1 + m)SI, E_2(t) \leq \alpha(t)SI + \varepsilon E_1, I(t) \leq \beta(t)E_1. \quad (65)$$

Next, for any $t \geq 0$, there is a positive constant M , which is satisfied when $|X(t)| \leq M$; therefore,

$$-L(t; X(t), \alpha; \beta) = \frac{c_1\alpha^2(t) + c_2\beta^2(t)}{2} - E_1(t) - I(t) \geq d_1(|\alpha|^2 + |\beta|^2)^\rho - 2M. \quad (66)$$

Let $d_1 = \min\{\frac{c_1}{2}, \frac{c_2}{2}\}$, $d_2 = 2M$ and $\rho = 2$. Then, condition (v) is established. Hence, the optimal control can be realized.

Theorem 6.2. *There exist adjoint variables $\delta_1, \delta_2, \delta_3, \delta_4$ for the optimal control pair (α^*, β^*) that satisfy*

$$\begin{cases} d\delta_1(t) = \left[\alpha(t)(\delta_1 - \delta_2)(1+m)I + \alpha(t)(\delta_1 - \delta_3)I \right] dt - 2\lambda_1 dW_1, \\ d\delta_2(t) = \left[1 + \beta(t)(\delta_2 - \delta_4) + \sigma_2(\lambda_2 - \lambda_4) + \delta_2(\gamma_1 + \varepsilon + \mu) \right] dt + \lambda_2 dW_1 - \lambda_2 dW_2, \\ d\delta_3(t) = \left[\delta_3(\gamma_2 + \mu) \right] dt + \lambda_3 dW_1, \\ d\delta_4(t) = \left[\frac{1 - \alpha(t)(\delta_1 - \delta_2)(1+m)S + \alpha(t)(\delta_1 - \delta_3)S}{+\delta_4(\lambda + \mu) + \sigma_1(\lambda_1 - \lambda_2)(1+m)S + \sigma_1(\lambda_1 - \lambda_3)S} \right] dt + \lambda_4 dW_2. \end{cases} \tag{67}$$

The boundary conditions are as follows:

$$\delta_1(t_f) = \delta_2(t_f) = \delta_3(t_f) = \delta_4(t_f) = 0. \tag{68}$$

In addition, the optimal control pair (α^*, β^*) of the state system (Eq. 59) can be given by

$$\begin{aligned} \alpha^*(t) &= \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\}, \\ \beta^*(t) &= \min \left\{ 1, \max \left\{ 0, \frac{(\delta_2 - \delta_4)E_1}{c_2} \right\} \right\}. \end{aligned} \tag{69}$$

Proof. In order to obtain the expression of the optimal control system and optimal control pair, we define a Hamiltonian function, which can be written as

$$\begin{aligned} H = & -E_1(t) - I(t) + \frac{c_1}{2}\alpha^2(t) + \frac{c_2}{2}\beta^2(t) + \delta_1[B - \alpha(t)(1+m)SI - \alpha(t)SI - \mu S] \\ & + \delta_2[\alpha(t)(1+m)SI - \beta(t)E_1 - \gamma_1 E_1 - \varepsilon E_1 - \mu E_1] + \delta_3[\alpha(t)SI + \varepsilon E_1 - \gamma_2 E_2 - \mu E_2] \\ & + \delta_4[\beta(t)E_1 - \lambda I - \mu I] + [-\lambda_1 \sigma_1(2+m)SI] + [\lambda_2(\sigma_1(1+m)SI - \sigma_2 E_1)] \\ & + \lambda_3 \sigma_1 SI + \lambda_4 \sigma_2 E_1. \end{aligned} \tag{70}$$

According to the Pontryagin maximum principle, the adjoint system can be written as

$$\frac{d\delta_1}{dt} = -\frac{\partial H}{\partial S}, \frac{d\delta_2}{dt} = -\frac{\partial H}{\partial E_1}, \frac{d\delta_3}{dt} = -\frac{\partial H}{\partial E_2}, \frac{d\delta_4}{dt} = -\frac{\partial H}{\partial I}, \tag{71}$$

and the boundary conditions of the adjoint system are

$$\delta_1(t_f) = \delta_2(t_f) = \delta_3(t_f) = \delta_4(t_f) = 0. \tag{72}$$

The optimal control formula can be written as

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= c_1 \alpha - \delta_1(1+m)SI - \delta_1 SI + \delta_2(1+m)SI + \delta_3 SI = 0, \\ \frac{\partial H}{\partial \beta} &= c_2 \beta - \delta_2 E_1 + \delta_4 E_1 = 0. \end{aligned} \tag{73}$$

Then, the optimal control pair (α^*, β^*) can be calculated based on Eq. 73 as

$$\begin{aligned} \alpha^*(t) &= \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\}, \\ \beta^*(t) &= \min \left\{ 1, \max \left\{ 0, \frac{(\delta_2 - \delta_4)E_1}{c_2} \right\} \right\}. \end{aligned} \tag{74}$$

Remark 6.1: So far, the optimal control system that can be obtained includes the state system (Eq. 59) with the initial conditions $S(0) = S_0, E_1(0) = E_{1,0}, E_2(0) = E_{2,0}, I(0) = I_0$ and the adjoint system (Eq. 67) with boundary conditions with the optimization conditions. The optimal control system can be written as

$$\begin{cases} dS(t) = \left[B - \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (1+m)SI \right. \\ \left. - \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} SI - \mu S \right] dt \\ - \sigma_1(1+m)SI dW_1(t) - \sigma_1 SI dW_2(t), \\ dE_1(t) = \left[\min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (1+m)SI \right. \\ \left. - \min \left\{ 1, \max \left\{ 0, \frac{(\delta_2 - \delta_4)E_1}{c_2} \right\} \right\} E_1 - \gamma_1 E_1 - \varepsilon E_1 - \mu E_1 \right] dt \\ + \sigma_1(1+m)SI dW_1(t) - \sigma_2 E_1 dW_2(t), \\ dE_2(t) = \left[\min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} SI \right. \\ \left. + \varepsilon E_1 - \gamma_2 E_2 - \mu E_2 \right] dt \\ + \sigma_1 SI dW_1(t), \\ dI(t) = \left[\min \left\{ 1, \max \left\{ 0, \frac{(\delta_2 - \delta_4)E_1}{c_2} \right\} \right\} E_1 - \lambda I - \mu I \right] dt + \sigma_2 E_1 dW_2(t), \\ d\delta_1(t) = \left[\min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (\delta_1 - \delta_2)(1+m)I \right. \\ \left. + \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (\delta_1 - \delta_3)I \right] dt \\ + \sigma_1(\lambda_1 - \lambda_2)(1+m)I - \lambda_1 \sigma_1 I + \mu \delta_1 \\ - 2\lambda_1 dW_1, \\ d\delta_2(t) = \left[1 + \min \left\{ 1, \max \left\{ 0, \frac{(\delta_2 - \delta_4)E_1}{c_2} \right\} \right\} (\delta_2 - \delta_4) \right. \\ \left. + \sigma_2(\lambda_2 - \lambda_4) + \delta_2(\gamma_1 + \varepsilon + \mu) \right] dt + \lambda_2 dW_1 - \lambda_2 dW_2, \\ d\delta_3(t) = [\delta_3(\gamma_2 + \mu)] dt + \lambda_3 dW_1 \\ d\delta_4(t) = \left[1 - \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (\delta_1 - \delta_2)(1+m)S \right. \\ \left. + \min \left\{ 1, \max \left\{ 0, \frac{[\delta_1(2+m) - \delta_2(1+m) - \delta_3]SI}{c_1} \right\} \right\} (\delta_1 - \delta_3)S \right] dt \\ + \delta_4(\lambda + \mu) + \sigma_1(\lambda_1 - \lambda_2)(1+m)S + \sigma_1(\lambda_1 - \lambda_3)S \\ + \lambda_4 dW_2, \end{cases} \tag{75}$$

and

$$\delta_1(t_f) = \delta_2(t_f) = \delta_3(t_f) = \delta_4(t_f) = 0. \tag{76}$$

7 Numerical simulations

In this section, numerical simulation is conducted using the Runge–Kutta algorithm to validate the theorem of the stochastic system (2). Since the values of the parameters are not explicitly given in most recent studies, this section determines reasonable values for the model’s parameters by combining the value range of the basic reproduction number R_0 and the basic conditions proposed by the theorem.

To observe the effect of external environmental factors on information spread and the effect of stochastic perturbations on changes in population characteristics in the deterministic model, the parameter values must satisfy the fundamental condition that information can spread in the social system, i.e., the basic regeneration number $R_0 > 1$. Therefore, the parameter value can be $B = 1, \alpha = 0.5, m = 0.3, \mu = 0.3, \gamma_1 = 0.5, \gamma_2 = 0.7, \beta = 0.6, \varepsilon = 0.2, \lambda = 0.2$.

First, the perturbation strength can be $\sigma = 0.001$. Figure 2 shows the probability histogram of population $S(t), E_1(t), E_2(t), I(t)$. As shown in Figure 2, the probability of all populations obeying the social system is stable. Figure 3 compares the density change trend

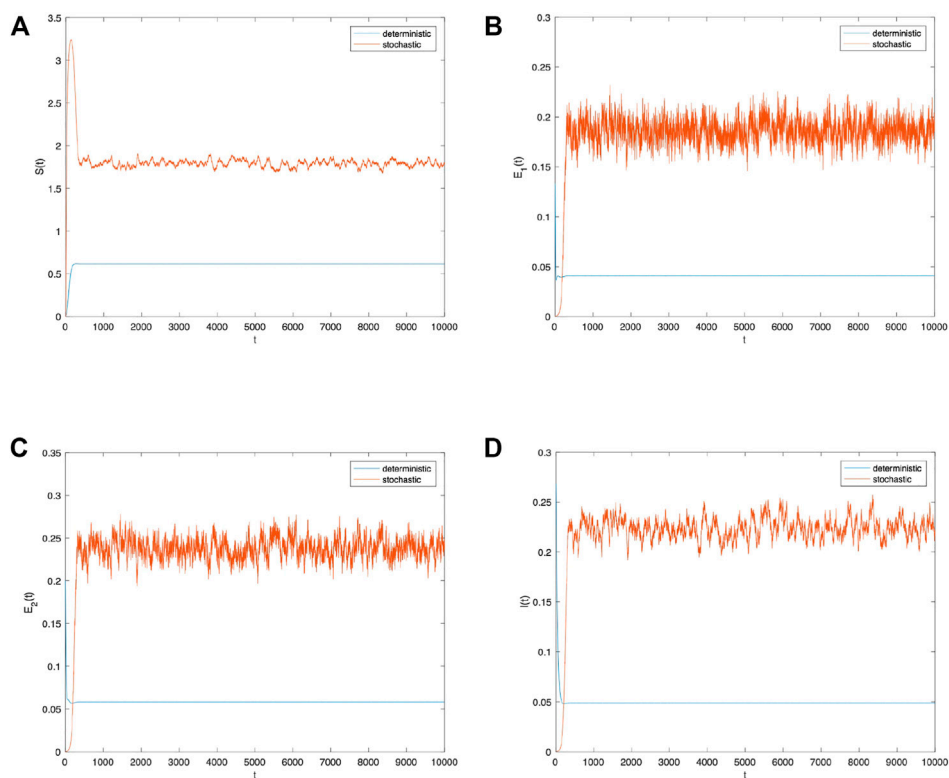


FIGURE 3 Comparison between the deterministic model and stochastic model of the densities of (A) $S(t)$, (B) $E_1(t)$, (C) $E_2(t)$, and (D) $I(t)$ change over time when $\sigma_i(i=1,2)=0.001$.

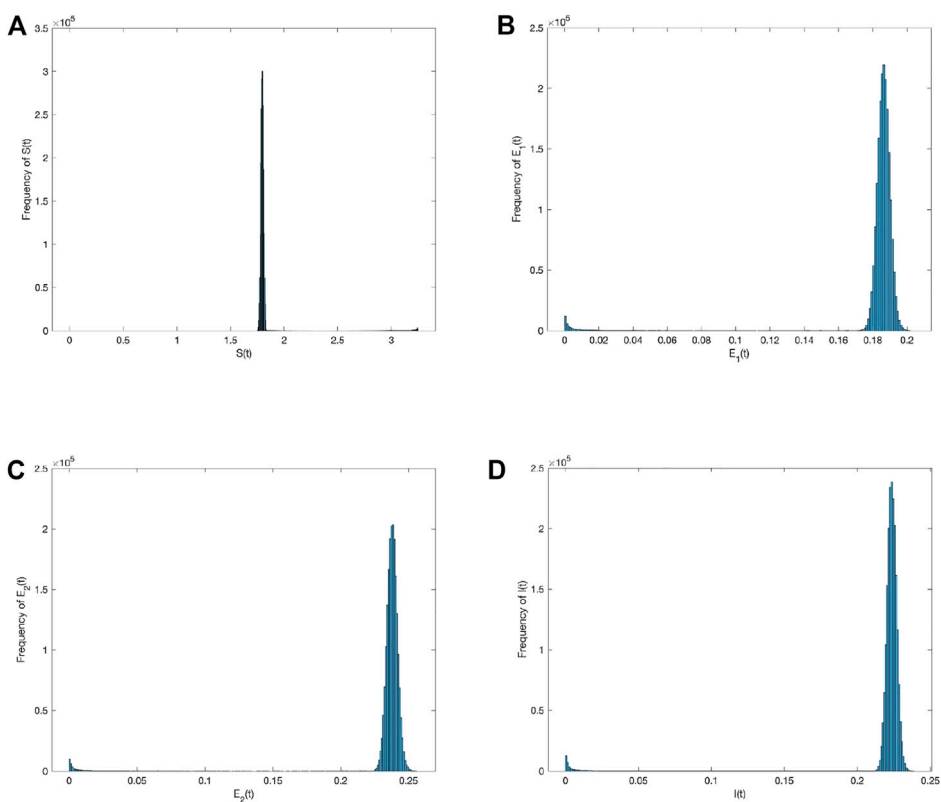


FIGURE 4 Frequency histograms of (A) $S(t)$, (B) $E_1(t)$, (C) $E_2(t)$, and (D) $I(t)$ when $\sigma_i(i=1,2)=0.0001$.

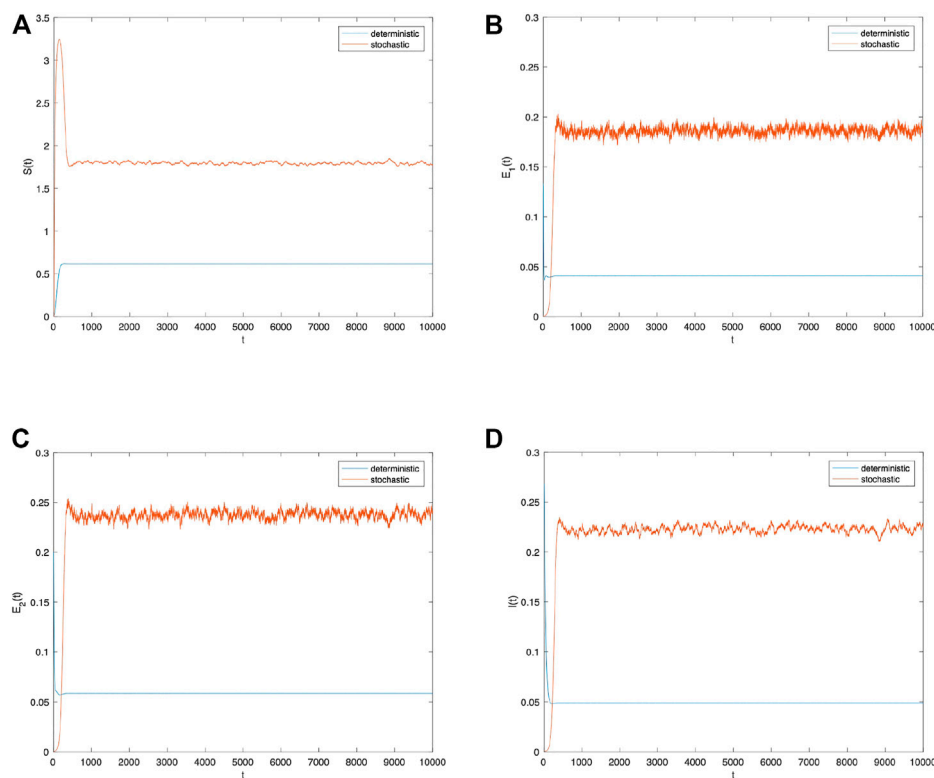


FIGURE 5

Comparison between the deterministic model and stochastic model of the densities of (A) $S(t)$, (B) $E_1(t)$, (C) $E_2(t)$, and (D) $I(t)$ change over time when $\sigma_i(i=1,2)=0.0001$.

between deterministic and stochastic systems about population $S(t)$, $E_1(t)$, $E_2(t)$, $I(t)$ over time. As shown in Figure 3, information spreads significantly better in the system with the addition of stochastic perturbation than in the deterministic system, indicating that stochastic environmental perturbations play a positive role in information spread. However, the information is in an unstable state as it spreads through the social system, and the population density of each group constantly fluctuates over time.

Second, the perturbation strength σ is reduced to 0.0001, while all other parameters remain unchanged. Figure 4 depicts the population $S(t)$, $E_1(t)$, $E_2(t)$, $I(t)$ probability histogram. Compared to Figure 2, the probabilities obeyed by all populations in the social system are stable, and their distributions have converged in Figure 4. Figure 5 compares the density change trend between deterministic and stochastic systems about population $S(t)$, $E_1(t)$, $E_2(t)$, $I(t)$ over time. Even though the perturbation intensity decreases in Figure 5, the stochastic system has greater information spread than the deterministic system. The information spread throughout the social system is not sufficiently stable as it is subject to fluctuations. In other words, information spread in a stochastic system produces fluctuations regardless of changes in perturbation strength. Due to the addition of random environmental factors, the densities of all populations are greater than those spread in a deterministic system.

Next, to observe the effects of different perturbation strengths on information spread, the trend plots of information spread over time for the stochastic system with perturbation strengths of 0.001 and

0.0001, respectively, are combined to be analyzed. As shown in Figure 6, as the intensity of the perturbation decreases, the fluctuation of information spread tends to become smoother, indicating that information spreads more easily in systems with random environmental factors and that controlling the random factors in the system can control the fluctuation of information spread effectively.

Finally, to verify the effect of the control strategy proposed in this chapter, the author kept other parameters constant and observed the density trends of the population $E_1(t)$ and $I(t)$ over time when the optimal control strategy was used by controlling the stochastic parameters α and β . As shown in Figure 7, when the perturbation strength $\sigma = 0.001$, the densities of populations $E_1(t)$ and $I(t)$ with the optimal control strategy for stochastic parameters α and β are superior to those without a control strategy. As shown in Figure 8, optimal control has a positive effect on information dissemination when the perturbation intensity $\sigma = 0.0001$ is reduced further. Furthermore, change the values of other parameters and compare the densities of $E_1(t)$ and $I(t)$ change trends with different control strategies under different intensities of perturbation. As shown in Figure 9, no matter how the parameter values change, the optimal control strategy proposed in this paper can promote information dissemination effectively. At the same time, adopting optimal control strategies can suppress the fluctuations of information during its spreading.

According to the results of theoretical analysis and numerical simulation, random factors can promote the

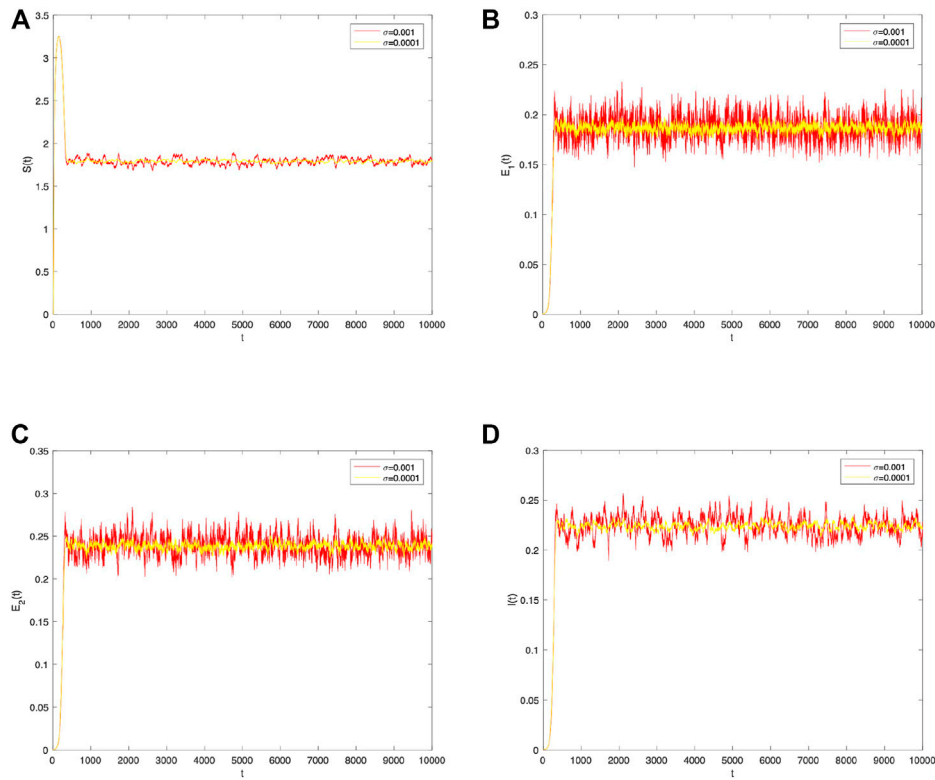


FIGURE 6 Comparison between $\sigma_i(i=1,2)=0.001$ and $\sigma_i(i=1,2)=0.0001$ of the densities of (A) $S(t)$, (B) $E_1(t)$, (C) $E_2(t)$, and (D) $I(t)$ change over time.

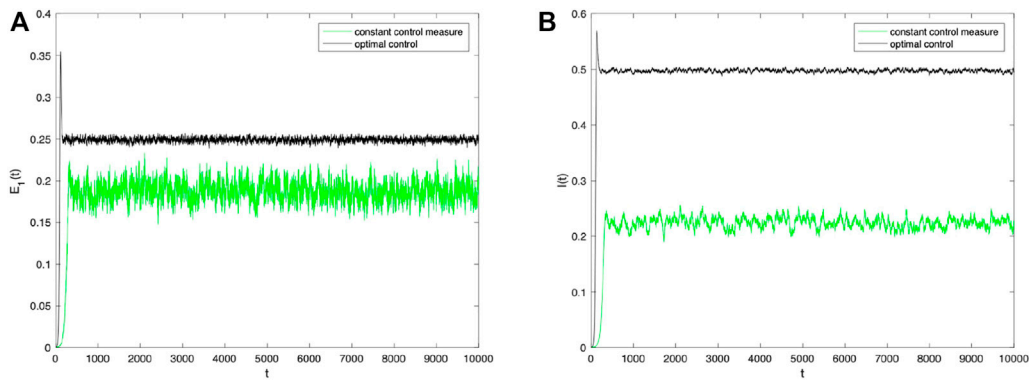


FIGURE 7 Densities of (A) $E_1(t)$ and (B) $I(t)$ change over time when $\sigma_i(i=1,2)=0.001$ under constant control measure and optimal control.

dissemination of information. For positive information, it is necessary to fully utilize the random phenomena in the social system to spread beneficial information for social development. For negative information, it is necessary to minimize the interference of random factors on the information to suppress the dissemination of information that is harmful to social development. Technological innovation information plays an important role in the development of enterprises in practical applications [3]. It is usually necessary to leverage the

randomness of the social system for technological innovation information, enhance the range of exposure to technological innovation information in society, and promote the random dissemination of technological innovation information among different individuals so as to maximize the diffusion effect. On the contrary, the spread of rumor information hinders the development of society. During the outbreak of COVID-19 all over the world, the rumor that Shuanghuanglian oral liquid could treat COVID-19 was widely spread in China [39], which

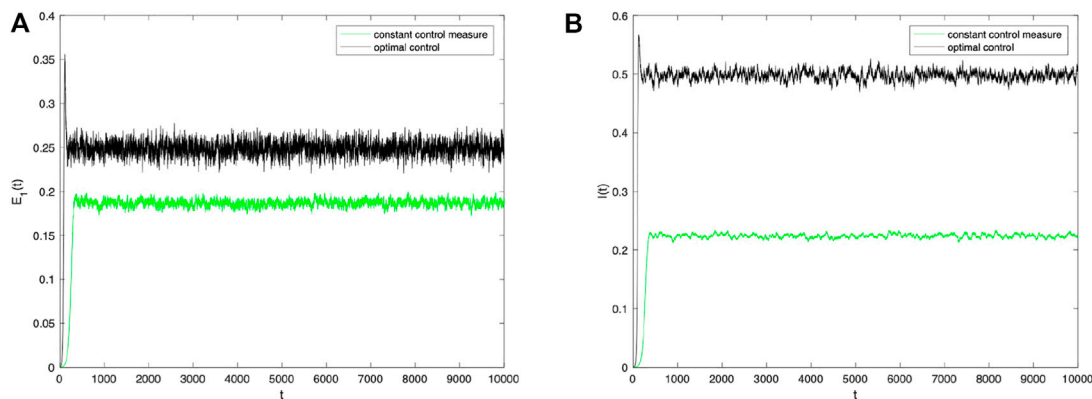


FIGURE 8 Densities of (A) $E_1(t)$ and (B) $I(t)$ change over time when $\sigma_i(i=1,2)=0.0001$ under constant control measure and optimal control.

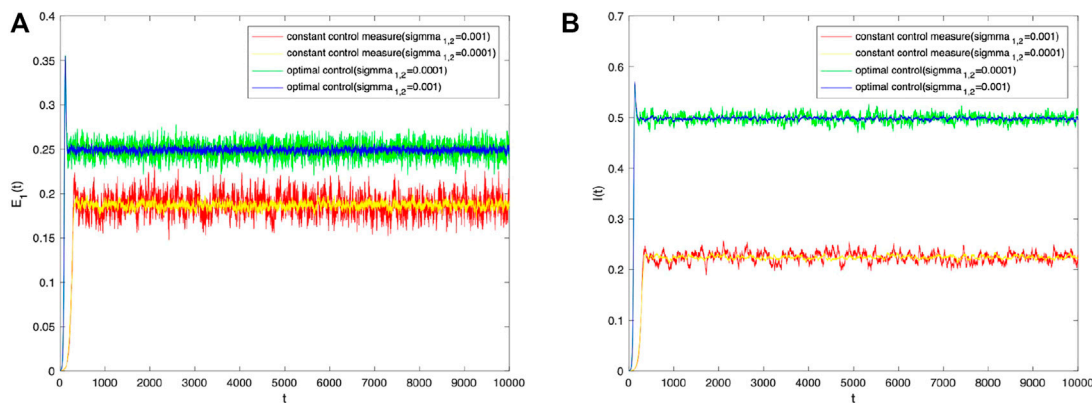


FIGURE 9 Densities of (A) $E_1(t)$ and (B) $I(t)$ change over time with different intensities of perturbation under constant control measure and optimal control.

led to the citizens rushing to purchase the drug and caused social panic. It can be seen from this incident that the continuous emergence of We-media and unscrupulous merchants is a random phenomenon, and effective control of these random phenomena can quickly suppress the spread of rumors.

8 Conclusion

In this paper, stochastic factors in social systems are added to the deterministic model. Then, the stochastic *S2EIR* model with parameter perturbations is proposed, with two parameters represented by Gaussian white noise: the contact rate of super-spreaders and the conversion rate of information spreaders. The existence of a global positive solution is demonstrated, and sufficient conditions for information disappearance and smooth distribution of information are calculated. In addition, the optimal control strategy for the stochastic model is proposed. The effect of the probability density distribution and white noise perturbation on

information dissemination is validated through numerical simulation. The dissemination trends of the information with different perturbation strengths are compared.

From the research presented in this paper, the following conclusions can be drawn: 1) White noise perturbation can promote information dissemination, and random environmental factors play a positive role in information dissemination. 2) As the intensity of perturbation increases, the randomness of the model and the fluctuation of the information spread trend become more obvious. 3) Information spread can be effectively controlled by controlling the stochastic parameters, and the optimal control strategy of this paper is based on the optimal values calculated by the control variables, which is different from previous studies.

There are many uncertainties in the actual social system, and it is more scientific to incorporate these uncertainties into deterministic models in order to develop stochastic models of dissemination. In this paper, the author considers the information spread phenomena of super-spreaders and implicit exposers and uses random perturbation terms to analyze the method of controlling

information spread in the actual social system. The results indicate that for positive information, the randomness and complexity of the social system should be utilized to increase the spread of information. In contrast, for negative information, randomness in the social system should be suppressed to the greatest extent possible to limit information dissemination.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

Conceptualization: SK and XH; methodology: SK and YH; software: SK and YH; validation: SK, XH, and HL; formal analysis: SK and YH; investigation: SK and YH; data curation: SK and YH; writing—original draft preparation: SK; and writing—review and editing: XH and HL. All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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