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Fractional solitons: New phenomena and exact solutions

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The fractional solitons have demonstrated many new phenomena, which cannot be explained by the traditional solitary wave theory. This paper studies some famous fractional wave equations including the fractional KdV–Burgers equation and the fractional approximate long water wave equation by a modified tanh-function method. The solving process is given in details, and new solitons can be rigorously explained by the obtained exact solutions. This paper offers a new window for studying fractional solitons.

KEYWORDS

time-fractional KdVB model, fractional approximate long water wave model, exact solutions, fractional solitons, fractional complex transform

1 Introduction

A fractional solitary wave [1] has some special properties which cannot be explained by the traditional soliton theory. The traditional soliton is a single wave with the same shape in propagation, while the fractional soliton has some amazing memory and non-local properties, which means the present wave morphology depends upon its history. This is caused by the intrinsic property of the fractional derivative [2]. The fractal solitary waves, on the other hand, are waves traveling along an unsmooth boundary [3, 4]. And the fractal solitary wave has the local property, the unsmooth boundary affects its wave shape. Here, the two-scale fractal theory [5, 6] is adopted to figure out the basic property of the unsmooth boundary.

This paper focuses on fractional solitons, which can describe physical phenomena more accurately and reflect their intrinsic properties deeply. Therefore, fractional solitons have attracted increasing attention from both physics and oceanography. For example, shallow water waves [7, 8] can describe the effects of waves in the ocean better than other mathematical models. Shallow water waves are fluctuations in the ocean with wavelengths much greater than the depth of the water (usually more than 25 times), and the dispersion of water waves is one of the key properties in many shallow water wave models, which has obvious memory property. Fractional shallow water equations can describe the propagation of waves in dispersed media and model the hydrodynamics of lakes, estuaries, tidal stalls, and coastal waves, as well as deep-ocean tides. These fractional differential equations have a significant impact on the study of fluid motion in ocean waves and the soliton theory as well; however, a serious bottleneck was hit, that is, the fractional model is extremely difficult to be solved analytically. Therefore, many scholars focused on using different methods to find fractional solitons. For instance, the first integral method [9], the fractional sub-equation method [10], the homotopy perturbation method [11-13] and its modifications, Mohand transform–homotopy perturbation method [14, 15], two-scale transform–-homotopy perturbation method [16], Laplace transform–homotopy perturbation method [17], Li–He's modified homotopy perturbation method [18-20], the tanh-function method [21, 22] and its modification—tanh function expansion method [23]—and modified extended tanh-function method [24,25]. It is worth mentioning that fractional complex transform was first proposed by [26]; it can convert fractional differential equations directly into ordinary differential equations. This method makes a significant contribution to finding exact solutions of fractional differential equations, and it was applied to gain insights into physical properties of the time-fractional Schrodinger equation [27] and the time-fractional Camassa–Holm equation [28].

In the current article, our concern is to find some exact solutions of the following two non-linear FPDEs *via* the modified extended tanh-function method with the fractional complex transform.

1) The time-fractional KdV–Burgers (KdVB) equation of the form [29

$$\frac{\partial^\eta u}{\partial t^\eta} + wu\frac{\partial u}{\partial x} + \rho\frac{\partial^2 u}{\partial x^2} + s\frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where w, ρ, and s are real constants and $0 < \eta < 1$. The KdVB equation ($\eta = 1$) is a well-known mathematical model for describing waves on shallow water surfaces; it plays an essential role in both applied mathematics and physics. This equation can be used to describe and analyze a few foremost physical contents related to liquids, dispersion, viscosity, and wave dynamics. For example, it is used to study the spread of waves in elastic tubes filled with viscous fluids [30] and to analyze the propagation of wave-like pores in shallow water [31].

However, with the increasing irregularities and non-linearities in wave motion observed by other scholars, the broader outlook establishment for this model is necessary. Therefore, an increasing number of scholars began to study the extended classical model into a new model with time-fractional derivatives to deal with what the traditional KdVB equation ($\eta = 1$) cannot do.

There have been some common methods to solve fractional KdVB equations. For instance, [32] extended the homotopy perturbation method to solve time-space fractional equations. [29] applied the residual power series method (RPSM) for finding approximate solutions of the time-fractional KdVB equation. [33] solved the time-fractional KdVB equation numerically by the Petrov–Galerkin method.

2) The fractional approximate long water wave equation is given as [34]

$$\begin{aligned} &D_t^\eta u - uD_x^\eta u - D_x^\eta v + aD_x^{2\eta} u = 0, \\ &D_t^\eta v - D_x^\eta (uv) - aD_x^{2\eta} v = 0, \end{aligned} \tag{2}$$

where $0 < \eta < 1$ and a is a real parameter. As a famous equation to describe the propagation of shallow water waves, it is also important for its amazing fractional solitons, so its exact solutions are much needed to gain insights deeply into the properties of the fractional solitary waves. Up to now, some explicit solutions appeared in the literature, for instance, [34] found three traveling wave solutions by the fractional sub-equation method, [35] obtained an exact solution by using the (G'/G)-method, and [36] also constructed an exact solution by the generalized Kudryashov method. Although much achievement was obtained, its full breathtaking panorama has not been offered yet.

The article is divided into the following sections: First, an introduction is given to the basic knowledge in Section 2; second, in Section 3, the general steps for the solution are given in detail; and finally, the applications and the conclusions are organized in Section 4 and Section 5, respectively.

2 Preliminaries

Regarding the definition of fractional derivatives, many mathematicians started from different perspectives and gave different definitions. Here are some definitions.

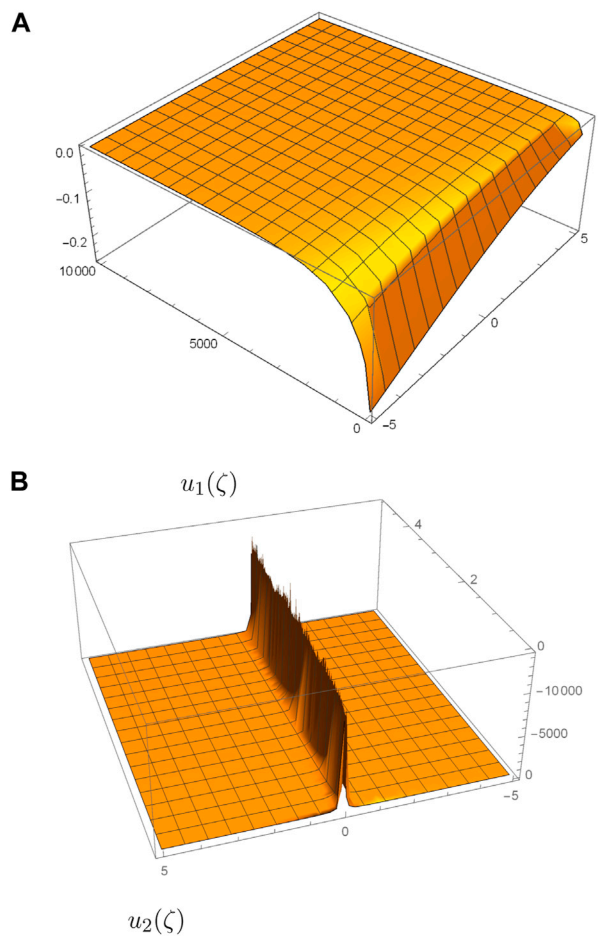

FIGURE 1
Three-dimensional plots of u1(ζ) and u2(ζ) in case 1 for η=0.5, w=1, ρ=1, s=1, l=1, L=1.

1) Caputo fractional derivative [37, 38]:

$$D_x^\eta [f(x)] = \frac{1}{\Gamma(n-\eta)}\int_0^x (x-t)^{n-\eta-1}\frac{d^n f(t)}{dt^n}dt. \tag{3}$$

2) Jumarie's modified Riemann–Liouville (R–L) fractional derivative [39]:

$$D_t^\eta g(t) = \begin{cases} \frac{1}{\Gamma(1-\eta)}\int_0^t (t-\xi)^{-\eta-1}(g(\xi)-g(0))d\xi, & \eta<0, \\ \frac{1}{\Gamma(1-\eta)}\frac{d}{dt}\int_0^t (t-\xi)^{-\eta}(g(\xi)-g(0))d\xi, & 0<\eta<1, \\ \left(g^{(n)}(t)\right)^{(\eta-n)}, & n\le\eta<n+1, n\ge 1, \end{cases} \tag{4}$$

where

$$\Gamma(\eta) = \int_0^{+\infty} x^{\eta-1}e^{-x}dx, \ \eta>0. \tag{5}$$

(3) He's fractional derivative [20, 40]:

$$D_t^\eta f = \frac{1}{\Gamma(n-\eta)}\frac{d^n}{dt^n}\int_{t_o}^t (s-t)^{n-\eta-1}[f_0(s)-f(s)]ds, \tag{6}$$

where $f_0(x)$ is a known function.

4) Two-scale fractal derivative [41, 42]:

$$\frac{\partial w}{\partial t^{\eta}}(t^{\eta}) = \Gamma(1+\eta) \lim_{t-t_0=\triangle t, \triangle \neq 0} \frac{w(t^{\eta}) - w(t_0^{\eta})}{(t-t_0)^{\eta}}, \tag{7}$$

where $\triangle t$ is the period required for the motion through a gap of a porous space.

In addition, there are other famous derivatives in the literature such as the Atangana–Baleanu derivative with non-local and non-singular kernel [43, 44]. In this paper, we adopt the Jumarie's modified R–L derivative definition. Some of its important properties are as follows:

$$D_t^{\eta} t^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\eta)} t^{m-\eta}, \tag{8}$$

$$D_t^{\eta}(cg(x)) = cD_t^{\eta} g(x),\ c \text{ is a constant}, \tag{9}$$

$$D_t^{\eta}\{g(\omega) + f(\omega)\} = D_t^{\eta} g(\omega) + D_t^{\eta} f(w). \tag{10}$$

3 Basic idea of the modified tanh-function expansion method

Considering the following equation

$$P(u, D_t^{\eta}u, D_x^{\gamma}u, D_t^{\eta}D_t^{\eta}u, D_t^{\eta}D_x^{\gamma}u, D_x^{\gamma}D_x^{\gamma}u, \cdots) = 0, \quad (0 < \eta, \gamma < 1), \tag{11}$$

where $D_t^{\eta}u, D_x^{\gamma}u, D_t^{\eta}D_t^{\eta}u, D_t^{\eta}D_x^{\gamma}u\cdots$ are the modified R–L fractional derivatives. P presents the polynomial function. To solve this equation, by using the modified tanh-function expansion method, we divide the solution processes into three steps.

Step 1: Using the fractional complex transformation [26, 45]

$$u(x,t) = u(\zeta), \quad \zeta = \frac{lx^{\gamma}}{\Gamma(\gamma+1)} + \frac{kt^{\eta}}{\Gamma(\eta+1)}, \tag{12}$$

where l and k are constants and $l, k \neq 0$. By the chain rule [45],

$$D_t^{\eta}u = \sigma_t \frac{\partial u(\zeta)}{d\zeta} D_t^{\eta}\zeta, \qquad D_x^{\eta}u = \sigma_x \frac{\partial u(\zeta)}{d\zeta} D_x^{\eta}\zeta,$$
$$D_t^{2\eta}u = (\sigma_t)^2 \frac{\partial^2 u(\zeta)}{d\zeta^2} D_t^{2\eta}\zeta, \qquad D_x^{2\eta}u = (\sigma_x)^2 \frac{\partial^2 u(\zeta)}{d\zeta^2} D_x^{2\eta}\zeta, \tag{13}$$

where σ_t and σ_x are sigma indices. We take $\sigma_t = \sigma_x = L$, where L is a constant. Then, substituting Eqs 12 and 13 into Eq. 11, we obtain a non-linear ODE that contains only variable ζ:

$$D(u, u', u'' \cdots) = 0, \tag{14}$$

where $u' = \frac{du}{d\zeta}, u'' = \frac{d^2u}{d\zeta^2}\cdots$ and D presents the polynomial function.

Step 2: Supposing Eq. 14 has the solution as Eq. 15

$$u(\zeta) = \sum_{i=0}^{n} a_i \Phi^i(\zeta), \tag{15}$$

where Φ is a function about ζ, and it satisfies the Riccati equation

$$\Phi' = \tau + \Phi^2, \tag{16}$$

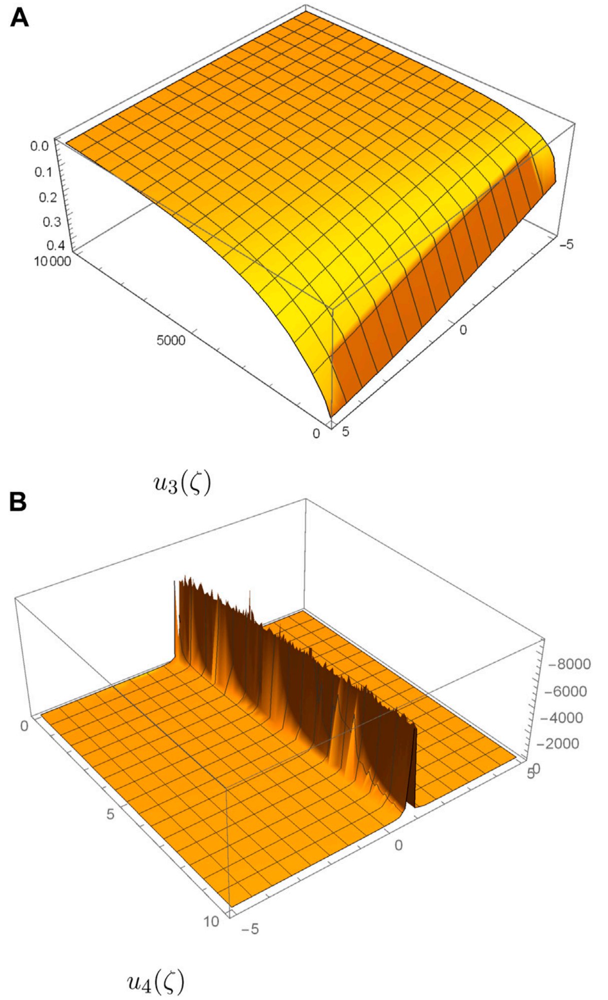

FIGURE 2
Three-dimensional plots of u3(ζ) and u4(ζ) in case 2 with η=0.5, w=1, ρ=1, s=1, l=1, L=1.

τ is a constant, and $a_i (i = 0, 1, 2, \ldots, n)$ are undetermined constant. n is a balancing parameter which is determined by the homogeneous balance method. Φ has the following three types of solutions according to the different values of constant τ

$$\begin{cases} \Phi = -\sqrt{-\tau}\tanh\sqrt{-\tau}\,\zeta, & \tau < 0, \\ \Phi = -\sqrt{-\tau}\coth\sqrt{-\tau}\,\zeta, & \tau < 0, \\ \Phi = \sqrt{\tau}\tan\sqrt{\tau}\,\zeta, & \tau > 0, \\ \Phi = \sqrt{\tau}\cot\sqrt{\tau}\,\zeta, & \tau > 0, \\ \Phi = -\frac{1}{\zeta}, & \tau = 0. \end{cases} \tag{17}$$

Step 3: Substituting Eq. 15 and 16 into Eq. 14, we obtain an iteration formulation to obtain the polynomial of Φ. Then, we get the algebraic equations about $a_i (i = 0, 1, 2, \ldots, n)$ and l, k, L, and τ by letting the coefficients of each power and constant terms of Φ to be 0. By solving them, we calculate the values of $a_i (i = 0, 1, 2, \ldots, n)$ and l, k, L, and τ. Thus, the exact solution of Eq. 11 is obtained from Eqs. 15–17.

4 Applications

We choose two different and classical equations named the time-fractional KdVB equation and the fractional approximate long water wave equation for applications. By the calculations of software, we obtain the exact solutions of these two equations and the 3D plots of the obtained solutions perform well.

4.1 Solving process for the fractional Kdv−Burgers model

Taking the fractional complex transform [26, 45]

$$\begin{aligned} u(x,t) &= u(\zeta),\\ \zeta &= lx + \frac{kt^{\eta}}{\Gamma(\eta+1)}. \end{aligned} \tag{18}$$

Then, the original equation Eq. (1) is converted into a non-linear ODE:

$$kLu' + lwuu' + l^2\rho u'' + l^3 su''' = 0. \tag{19}$$

Integrating once and the integral constant is equal to zero, Eq. 19 turns into

$$2kLu + lwu^2 + 2l^2\rho u' + 2l^3 su'' = 0, \tag{20}$$

where n is a balancing parameter. It is used to keep the balance between the term "u''" and the non-linear term "u^2"; we find $n = 2$. Therefore, Eq. 15 changed to

$$u(\zeta) = a_0 + a_1\Phi(\zeta) + a_2\Phi^2(\zeta). \tag{21}$$

Substituting Eqs 16 and 21 into Eq. 20, merging the terms of the same degree of Φ, and vanishing each coefficient of the resulted polynomials to zero, we obtain the equations for the unknowns a_0, a_1, a_2, l, k, L, and τ:

$$\begin{aligned} &2a_0kL + 2\tau l^2(a_1\rho + 2a_2\tau ls) + a_0^2 lw = 0,\\ &2a_2\tau l^2\rho + a_1(kL + 2\tau l^3 s + a_0 lw) = 0,\\ &a_1 l(2l\rho + a_1 w) + 2a_2(kL + 8\tau l^3 s + a_0 lw) = 0,\\ &2a_2 l\rho + 2a_1 l^2 s + a_1 a_2 w = 0,\\ &12 l^2 s + a_2 w = 0. \end{aligned} \tag{22}$$

Solving the aforementioned set of algebraic equations in the software application, the solutions of the original equation called four generalized hyperbolic function solutions are obtained.

Case 1.

$$a_0 = -\frac{3\rho^2}{25sw},\ a_1 = -\frac{12l\rho}{5w},\ a_2 = -\frac{12l^2 s}{w},\ k = \frac{6l\rho^2}{25Ls},\ \sigma = -\frac{\rho^2}{100l^2s^2},$$

which produces

$$u_1(\zeta) = -\frac{3\rho^2}{25sw} + \frac{6\rho^2}{25sw}\tanh\frac{\rho}{10ls}\zeta - \frac{3\rho^2}{25ws}\tanh^2\frac{\rho}{10ls}\zeta,\ \ \tau<0, \tag{23}$$

$$u_2(\zeta) = -\frac{3\rho^2}{25sw} + \frac{6\rho^2}{25sw}\coth\frac{\rho}{10ls}\zeta - \frac{3\rho^2}{25ws}\coth^2\frac{\rho}{10ls}\zeta,\ \ \tau<0, \tag{24}$$

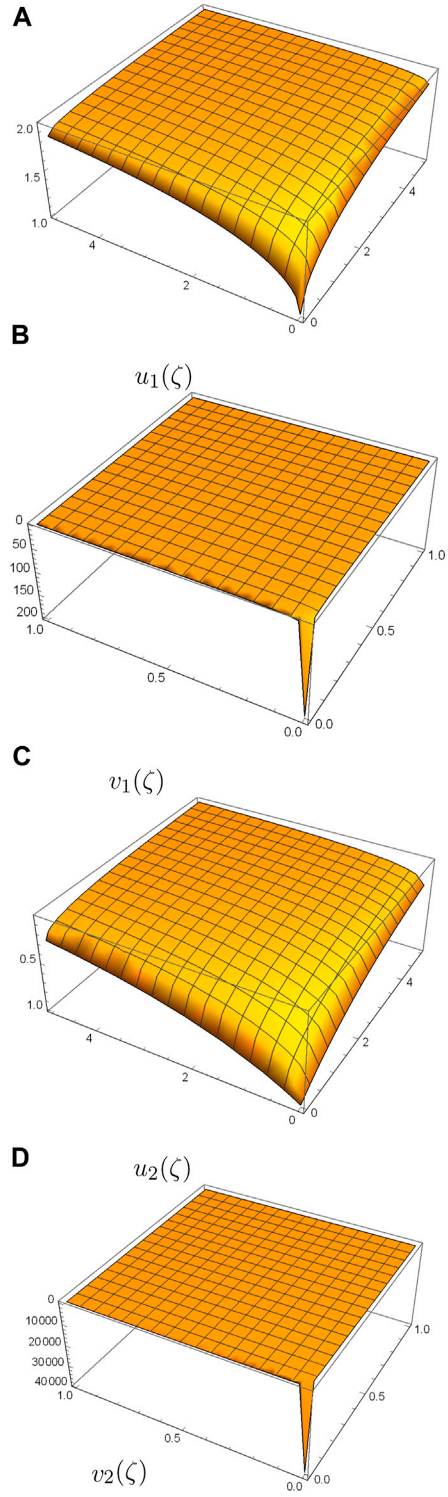

FIGURE 3
Three-dimensional plots of u1(ζ), v1(ζ)and u2(ζ), v2(ζ) of Eq. 2 for η=0.5, a=1, l=1, k=1, L=1.

where $\zeta = lx + \frac{kt^{\eta}}{\Gamma(\eta+1)}$, l is an arbitrary constant, and $l \neq 0$.

Figure 1 is the 3D plots of the obtained solutions of the KdVB equation in case 1 for $\eta = 0.5$, $w = 1$, $\rho = 1$, $s = 1$, $l = 1$, and $L = 1$.

Case 2.

$$a_0 = \frac{9\rho^2}{25sw},\ a_1 = -\frac{12l\rho}{5w},\ a_2 = -\frac{12l^2 s}{w},\ k = -\frac{6l\rho^2}{25Ls},\ \sigma = -\frac{\rho^2}{100l^2s^2},$$

which produces

$$u_3(\zeta) = \frac{9\rho^2}{25sw} + \frac{6\rho^2}{25sw}\tanh\frac{\rho}{10ls}\zeta - \frac{3\rho^2}{25ws}\tanh^2\frac{\rho}{10ls}\zeta,\ \tau<0, \quad (25)$$

$$u_4(\zeta) = \frac{9\rho^2}{25sw} + \frac{6\rho^2}{25sw}\coth\frac{\rho}{10ls}\zeta - \frac{3\rho^2}{25ws}\coth^2\frac{\rho}{10ls}\zeta,\ \tau<0, \quad (26)$$

where $\zeta = lx + \frac{kt^\eta}{\Gamma(\alpha+1)}$, l is an arbitrary constant, and $l \neq 0$.

Figure 2 shows the 3D plots of the obtained solutions of the KdVB equation in case 2 for $\eta = 0.5$, $w = 1$, $\rho = 1$, $s = 1$, $l = 1$, and $L = 1$.

4.2 Solving process for the fractional approximate long water wave equation

Equation 2 is transformed into the following ODEs by applying the fractional complex transformation Li and He [26] and He et al. [45]:

$$u(x,t) = u(\zeta),\ v(x,t) = v(\zeta),\quad \zeta = \frac{lx^\eta}{\Gamma(\eta+1)} + \frac{kt^\eta}{\Gamma(\eta+1)}. \quad (27)$$

Then, the following expressions are obtained:

$$kLu' - luLu' - lLv' + al^2L^2u'' = 0,\quad kLv' - lL(uv)' - al^2L^2v'' = 0. \quad (28)$$

We perform the same process as mentioned previously and we obtain

$$2ku - lu^2 - 2lv + 2al^2 Lu' = 0,\quad kv - l(uv) - al^2Lv' = 0. \quad (29)$$

Balancing "v" with "$u^{2\prime\prime}$ in the first equality in Eq. 29 and "v'" with "UV" in the second equality in Eq. 29, we find $n = 1$ and $m = 2$. Therefore, Eq. 15 can be written as

$$u(\zeta) = a_0 + a_1\Phi(\zeta),\quad v(\zeta) = b_0 + b_1\Phi(\zeta) + b_2\Phi^2(\zeta). \quad (30)$$

Substituting Eq. 16 and 30 into Eq. 29, merging the terms of the same degree of Φ, and making the coefficient of each item in the result equal to zero, we obtain the equations for the unknowns a_0, a_1, b_0, b_1, b_2, a, k, l, L, and τ

$$\begin{aligned} &2a_0k - a_0{}^2l + 2l(-b_0 + aa_1lL\tau) = 0,\\ &a_0a_1l + b_1l - a_1k = 0,\\ &a_1{}^2 + 2b_2 - 2aa_1lL = 0,\\ &-a_0b_0l + b_0k - ab_1l^2L\tau = 0,\\ &b_1k - a_0b_1l - a_1b_0l - 2ab_2l^2\ \ L\tau = 0,\\ &b_2k - a_0b_2l - a_1b_1l - ab_1l^2L = 0,\\ &a_1 + 2alL = 0. \end{aligned} \quad (31)$$

Solving the equations, we have

$$a_0 = \frac{k}{l},\ a_1 = -2alL,\ b_0 = \frac{k^2}{l^2},\ b_1 = 0,\ b_2 = -4a^2l^2L^2,\ \tau = -\frac{k^2}{4a^2l^4L^2}. \quad (32)$$

Finally, from Eqs 17, 27, 30 and 32, we obtain the following generalized hyperbolic function solutions of Eq. 2:

$$u_1(\zeta) = \frac{k}{l} + \frac{k}{l}\tanh\frac{k}{2al^2L}\zeta,\ \tau<0, \quad (33)$$

$$v_1(\zeta) = \frac{k^2}{l^2} - \frac{k^2}{l^2}\tanh^2\frac{k}{2al^2L}\zeta,\ \tau<0, \quad (34)$$

and

$$u_2(\zeta) = \frac{k}{l} + \frac{k}{l}\coth\frac{k}{2al^2L}\zeta,\ \tau<0, \quad (35)$$

$$v_2(\zeta) = \frac{k^2}{l^2} - \frac{k^2}{l^2}\coth^2\frac{k}{2al^2L}\zeta,\ \tau<0, \quad (36)$$

where $\zeta = \frac{lx^\eta}{\Gamma(\eta+1)} + \frac{kt^\eta}{\Gamma(\eta+1)}$, l and k are arbitrary constants, and $l, k \neq 0$.

Figure 3 shows the 3D plots of the obtained solutions of Eq. 2 for $\eta = 0.5$, $a = 1$, $l = 1$, $k = 1$, and $L = 1$.

5 Conclusion

In this paper, some attractive properties of the fractional solitons are elucidated through two examples, and this paper proposes a total new concept on the fractional soliton theory and gives a rigorous mathematical tool to gain deep insights into the physical properties of the fractional solitary solutions, which are practically applicable in many fields. Additionally, this paper also reveals the simplicity, comprehensibility, and effectiveness of the modified extended tanh-function method.

We anticipate that this paper offers a flood of opportunities for finding new physical phenomena of the fractional solitons, and this paper can be used as a good paradigm for future research.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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