



*- η -Ricci Soliton and Gradient Almost *- η -Ricci Soliton Within the Framework of Para-Kenmotsu Manifolds

Santu Dey¹ and Nasser Bin Turki^{2*}

¹Department of Mathematics, Bidhan Chandra College, Asansol, India, ²Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

The goal of the present study is to study the *- η -Ricci soliton and gradient almost *- η -Ricci soliton within the framework of para-Kenmotsu manifolds as a characterization of Einstein metrics. We demonstrate that a para-Kenmotsu metric as a *- η -Ricci soliton is an Einstein metric if the soliton vector field is contact. Next, we discuss the nature of the soliton and discover the scalar curvature when the manifold admits a *- η -Ricci soliton on a para-Kenmotsu manifold. After that, we expand the characterization of the vector field when the manifold satisfies the *- η -Ricci soliton. Furthermore, we characterize the para-Kenmotsu manifold or the nature of the potential vector field when the manifold satisfies the gradient almost *- η -Ricci soliton.

Keywords: Ricci flow, η -Ricci soliton, *- η -Ricci soliton, gradient almost *- η -Ricci soliton, para-Kenmotsu manifold

OPEN ACCESS

Edited by:

Josef Mikes,
Palacký University, Olomouc, Czechia

Reviewed by:

Simon Chiassi,
Fluminense Federal University, Brazil
Mića Stanković,
University of Nis, Serbia
Kanak Baishya,
Kurseong College, India

*Correspondence:

Nasser Bin Turki
nassert@ksu.edu.sa

Specialty section:

This article was submitted to
Mathematical and Statistical Physics,
a section of the journal
Frontiers in Physics

Received: 05 November 2021

Accepted: 03 January 2022

Published: 10 February 2022

Citation:

Dey S and Turki NB (2022) *- η -Ricci Soliton and Gradient Almost *- η -Ricci Soliton Within the Framework of Para-Kenmotsu Manifolds. *Front. Phys.* 10:809405. doi: 10.3389/fphy.2022.809405

1 INTRODUCTION

Consider a pseudo-Riemannian manifold (M, g) . We can say that the metric g is a Ricci soliton if there exists a smooth vector field V and a constant λ such that

$$\frac{1}{2} \mathcal{L}_V g + S + \lambda g = 0,$$

where L_V denotes the Lie derivative along V and S is the manifold's Ricci tensor. The vector field V is called the potential and λ is the soliton constant. Taking V to be zero, or a Killing vector, the condition reduces to the Einstein equation, and the soliton is called trivial.

A Ricci soliton is a self-similar solution of Hamilton's Ricci flow [1]: $\frac{\partial g(t)}{\partial t} = -2S(g(t))$, where $g(t)$ is a one-parameter family of metrics starting at $g(0) = g$. The potential V and the constant λ play a fundamental role in determining the soliton's nature. A soliton is called shrinking, steady, or expanding according to whether $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. If V is zero or Killing, then the Ricci soliton reduces to an Einstein manifold and the soliton is called a trivial soliton.

If the potential vector field V is the gradient of a smooth function f , denoted by Df then the soliton equation reduces to

$$\text{Hess} f + S + \lambda g = 0,$$

where $\text{Hess} f$ is Hessian of f . Perelman [2] proved that a Ricci soliton on a compact manifold is a gradient Ricci soliton.

In 2009, J. T. Cho and M. Kimura [3] introduced the concept of the η -Ricci soliton, which is another generalization of the classical Ricci soliton, and is given by the following:

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where μ is a real constant and η is a 1-form defined as $\eta(X) = g(X, \xi)$ for any $X \in \chi(M)$. Note that if $\mu = 0$, then the η -Ricci soliton reduces to a Ricci soliton.

In 2014, Kaimakamis and Panagiotidou [4] modified the definition of a Ricci soliton using the *-Ricci tensor S^* , which was introduced by Tachibana [5], in place of the Ricci tensor S . The *-Ricci tensor S^* is defined by

$$S^*(X, Y) = \frac{1}{2}(\text{trace}\{\phi \cdot R(X, \phi Y)\}),$$

where for all vector fields X and Y on M , ϕ is a (1,1) – tensor field. The aforementioned study used the concept of a *-Ricci soliton within the framework of real hypersurface of a complex space form.

In 2020, S. Dey et al. [6] defined the *-η-Ricci soliton as follows:

$$\mathcal{L}_\xi g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0.$$

To the authors’ knowledge, the results concerning the *-η-Ricci soliton were studied in the case in which the potential vector field V is the characteristic vector field ξ . Motivated from this, we generalized the definition by considering the potential vector field as an arbitrary vector field V and defined it as follows:

$$\mathcal{L}_V g + 2S^* + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.1}$$

where the manifold is considered as $(2n + 1)$ -dimensional. Next, if we consider the potential vector field V as the gradient of a smooth function f , then the *-η-Ricci soliton equation can be rewritten as follows:

$$\text{Hess} f + S^* + \lambda g + \mu\eta \otimes \eta = 0. \tag{1.2}$$

Here, the term “gradient almost *-η-Ricci soliton” denotes a gradient *-η-Ricci soliton for which we considered λ as a smooth function.

As for further literature study, Ricci solitons on paracontact geometry have been studied by many authors ([7–10]). In particular, Calvaruso and Perrone [9] explicitly studied Ricci solitons on 3-dimensional almost paracontact manifolds. In 2018, Ghosh and Patra [11] first studied the *-Ricci soliton on almost contact metric manifolds. Very recently, the *-Ricci soliton and its generalizations were investigated by Dey et al. [6, 12, 13, 15–21]. The case of the *-Ricci soliton in a para-Sasakian manifold was treated by Prakasha and Veerasha in the study mentioned in reference [22]. Furthermore, in 2019, V. Venkatesha et al. [23] considered the metric of an η-Einstein para-Kenmotsu manifold as a *-Ricci soliton and proved that the manifold is Einstein. In another study performed in 2019, I. K. Erken [24] considered Yamabe solitons on a 3-dimensional para-cosymplectic manifold and proved some vital results, including the fact that the manifold is either η-Einstein or Ricci flat. Several authors have also studied the η-Ricci soliton and its abstraction on paracontact metric manifolds; for instance, Dey et al. [25] considered a paracontact metric as a conformal Ricci soliton and a *-conformal Ricci soliton, Deshmukh et al. [26] studied certain results on Ricci

almost solitons, Sarkar et al. [27] examined a conformal η-Ricci soliton on a para-Sasakian manifold, and Naik et al. [28] considered a para-Sasakian metric as an η-Ricci soliton. In [29], Welyczko introduced the notion of the para-Kenmotsu manifold, which is analogous to the Kenmotsu manifold [30] in paracontact geometry; this topic was studied in detail by Zamkovoy [31]. Furthermore, Blaga studied certain aspects of η-Ricci solitons on para-Kenmotsu and Lorentzian para-Sasakian manifolds (see [32–34]). Motivated by these results, we considered a para-Kenmotsu metric as *-η-Ricci solitons and gradient *-η-Ricci solitons.

Based on the above facts and discussions in the research of contact geometry, a natural question arises:

Are there paracontact metric almost manifolds whose metrics are *-η-Ricci solitons?

In later sections, we have shown that the answer to this question is affirmative. The article is organized as follows: in **Section 2**, the basic definitions and facts about para-Kenmotsu manifolds are given. In the next section, we have explained that if the metric g represents a *-η-Ricci soliton and if the soliton vector field V is contact, then V is a strictly infinitesimal contact transformation and the manifold is Einstein. We have demonstrated some results in which a *-η-Ricci soliton admits a para-Kenmotsu manifold and determined the nature of the soliton and Laplacian of the smooth function. We have also considered that the manifold is η-Einstein when the manifold satisfies *-η-Einstein soliton and the vector field is conformal Killing. Furthermore, we have provided some examples to support our findings in that section. In **Section 4**, we have considered the gradient almost *-η-Ricci soliton and have shown that if the metric g represents a gradient almost *-η-Ricci soliton, then either M is Einstein or there exists an open set where the potential vector field V is pointwise collinear with the characteristic vector field ξ .

2 SOME PRELIMINARIES ON PARA-KENMOTSU MANIFOLDS

A $(2n + 1)$ -dimensional smooth manifold M is said to have an almost paracontact structure if it admits a vector field ξ , (1, 1)-tensor field ϕ , and 1-form η satisfying the following conditions:

$$i)\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$ii)\eta(\xi) = 1, \tag{2.2}$$

iii) ϕ induces an almost paracomplex structure \mathcal{P} on the $2n$ -dimensional distribution $\mathcal{D} \equiv \ker(\eta)$, that is, $\mathcal{P}^2 \equiv I_{\chi(M)}$ and the eigensubbundles \mathcal{D}^+ and \mathcal{D}^- , corresponding to the eigenvalues 1, -1 of \mathcal{P} , respectively, have equal dimension n ; hence, $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

As an example of the paracontact structure, consider $M = S^n \times S^n$, the product of two copies of unit sphere S^n . Let P be the product structure on M , which is defined by the following:

$$P(U, V) = (U, -V),$$

where (U, V) is a vector field on M . Then P satisfies the following:

$$P^2 = I, TrP = 0.$$

Moreover, with respect to the product of Riemannian metric g on M , we can see that P is a symmetric operator. Now, considering the product $\bar{M} = M \times S^1$, let ξ be the unit vector field globally defined on S^1 and let η be 1-form dual to ξ on \bar{M} . Then for a vector field E on \bar{M} , we have the following:

$$E = X + \eta(E)\xi,$$

where $X = (U, V)$ is a vector field on $M = S^n \times S^n$. Next, we defined a $(1, 1)$ tensor field ϕ on \bar{M} as follows:

$$\phi(E) = P(X), E = X + \eta(E)\xi.$$

Then it follows that

$$\phi^2(E) = P^2(X) = X = E - \eta(E)\xi,$$

and that $\phi(\xi) = P(0) = 0$ and $\eta \circ \phi = 0$. Hence, (ϕ, ξ, η) is an almost paracontact structure on the $(2n + 1)$ -dimensional manifold \bar{M} . If a manifold M with an almost paracontact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n + 1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

holds for any $X, Y \in \chi(M)$, then g is called a compatible metric and the manifold (M, ϕ, ξ, η, g) is called an almost paracontact metric manifold. If an almost paracontact metric manifold satisfies the following:

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

then the manifold is called an almost para-Kenmotsu manifold. The normality of an almost paracontact structure (M, ϕ, ξ, η) is equivalent to the vanishing of the $(1,2)$ -torsion tensor defined by $N_\phi(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$, where $[\phi, \phi]$ is the Nijenhuis torsion tensor of ϕ and is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + \phi X, \phi Y - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any $X, Y \in \chi(M)$. A normal almost para-Kenmotsu manifold is called a para-Kenmotsu manifold.

On a $(2n + 1)$ -dimensional para-Kenmotsu manifold, the following properties hold:

$$\phi(\xi) = 0, \tag{2.5}$$

$$\eta \circ \phi = 0, \tag{2.6}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.7}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.8}$$

$$Q\xi = -2n\xi, \tag{2.9}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)], \tag{2.12}$$

for any $X, Y \in \chi(M)$, where \mathcal{L} and ∇ are the operators of Lie differentiation and covariant differentiation of g , respectively. Q denotes the Ricci operator associated with the Ricci tensor S defined by $S(X, Y) = g(QX, Y)$ and R denotes the Riemannian curvature tensor.

A $(2n+1)$ -dimensional Kenmotsu metric manifold is said to be a η -Einstein Kenmotsu manifold if there exist two smooth functions a and b that satisfy the following relation:

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.13}$$

for all $X, Y \in \chi(M)$. Clearly, if $b = 0$, then η -Einstein manifold reduces to an Einstein manifold. Now, considering $X = \xi$ in the last equation and using 2.10, we have $a + b = -2n$. Contracting (2.13) over X and Y , we obtained $r = (2n + 1)a + b$, where r denotes the scalar curvature of the manifold. Solving these two, we have $a = (1 + \frac{r}{2n})$ and $b = -(2n + 1 + \frac{r}{2n})$. Using these values, we can rewrite 2.13 as follows:

$$S(X, Y) = \left(1 + \frac{r}{2n}\right)g(X, Y) - \left(2n + 1 + \frac{r}{2n}\right)\eta(X)\eta(Y). \tag{2.14}$$

3 *-η-RICCI SOLITON ON PARA-KENMOTSU MANIFOLDS

In this section, we have considered that the metric g of a $(2n + 1)$ -dimensional para-Kenmotsu manifold represents a *-η-Ricci soliton. We recalled some important lemmas relevant to our results.

Lemma 1. ([25]) The Ricci operator Q on a $(2n + 1)$ -dimensional para-Kenmotsu manifold satisfies the following:

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{3.1}$$

$$(\nabla_\xi Q)X = -2QX - 4nX, \tag{3.2}$$

for an arbitrary vector field X on the manifold.

Lemma 2. Venkatesha et al. [23] deduced the expression of the *-Ricci tensor for a para-Kenmotsu manifold as follows:

$$S^*(X, Y) = -S(X, Y) - (2n - 1)g(X, Y) - \eta(X)\eta(Y). \tag{3.3}$$

In addition, we plugged $X = e_i$ and $Y = e_i$ into the above equation, where e_i denotes a local orthonormal frame, and sum over $i = 1, 2, \dots, (2n + 1)$ to arrive at the following equation:

$$r^* = -r - 4n^2, \tag{3.4}$$

where r^* is the *-scalar curvature of M .

Theorem 1. Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If the metric g represents a *-η-Ricci soliton and if the soliton vector field V is contact, then V is a strictly infinitesimal contact transformation and the manifold is Einstein.

Proof 1. Because the metric g of the para-Kenmotsu manifold represents a *-η-Ricci soliton, both of Eqs. 1.1, 3.3 are satisfied. Combining these two, we have the following:

$$(\mathcal{L}_V g)(X, Y) = 2S(X, Y) - (2\lambda - 4n + 2)g(X, Y) - 2(\mu - 1)\eta(X)\eta(Y). \tag{3.5}$$

Taking the covariant derivative in the previous equation w.r.t. an arbitrary vector field Z and using 2.8, we obtained the following:

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = 2(\nabla_Z S)(X, Y) - 2(\mu - 1)\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}, \tag{3.6}$$

for all $X, Y, Z \in \chi(M)$. Again, from Yano [35], we have the following commutation formula:

$$(\mathcal{L}_V \nabla_Z g - \nabla_Z \mathcal{L}_V g - \nabla_{[V,Z]} g)(X, Y) = -g((\mathcal{L}_V \nabla)(X, Z), Y) - g((\mathcal{L}_V \nabla)(Y, Z), X),$$

where g is the metric connection, that is, $\nabla g = 0$. Thus, the above equation reduces to the following:

$$(\nabla_Z \mathcal{L}_V g)(X, Y) = g((\mathcal{L}_V \nabla)(X, Z), Y) + g((\mathcal{L}_V \nabla)(Y, Z), X), \tag{3.7}$$

for all vector fields X, Y , and Z on M . Combining (3.6) and (3.7) through a straightforward combinatorial computation and applying the symmetry of $(\mathcal{L}_V \nabla)$, the previous equation yields the following:

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X) - 2(\mu - 1)\{g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\},$$

for arbitrary vector fields X, Y , and Z on M . Using 3.1, 3.2, the previous equation yields the following:

$$(\mathcal{L}_V \nabla)(X, \xi) = 2QX + 4nX, \tag{3.8}$$

for all $X \in \chi(M)$. Now, covariantly differentiating this with respect to an arbitrary vector field Y , we obtained the following:

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) = 2(\nabla_Y Q)X - (\mathcal{L}_V \nabla)(X, Y) + \eta(Y) \times (2QX + 4nX). \tag{3.9}$$

We know that $(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z)$. In view of 3.9 in the previous relation, we acquired the following:

$$(\mathcal{L}_V R)(X, Y)\xi = 2\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + 2\eta(X)(QY + 2nY) - 2\eta(Y)(QX + 2nX), \tag{3.10}$$

for arbitrary vector fields X and Y on M . Setting $Y = \xi$ in the aforementioned equation and using 2.10, 3.1, 3.2, we obtained the following:

$$(\mathcal{L}_V R)(X, \xi)\xi = 0. \tag{3.11}$$

Now, taking 3.5 in account, the Lie derivative of $g(\xi, \xi) = 1$ along the potential vector field V yields the following:

$$\eta(\mathcal{L}_V \xi) = \lambda + \mu. \tag{3.12}$$

Plugging $Y = \xi$ and noting 2.2, Eq. 3.5 provides the following:

$$(\mathcal{L}_V \eta)X - g(X, \mathcal{L}_V \xi) = -(2\lambda + 2\mu)\eta(X), \tag{3.13}$$

which holds for an arbitrary vector field X on M . From 2.10, we computed $R(X, \xi)\xi = \eta(X)\xi - X$. Taking the Lie derivative along the potential vector field V and inserting 3.12 and 3.13 in account, this reduces to the following:

$$(\mathcal{L}_V R)(X, \xi)\xi = 2(\lambda + \mu)(X - \eta(X)\xi), \tag{3.14}$$

for all $X \in \chi(M)$. Finally, comparing 3.1 and 3.14, we have $2(\lambda + \mu)(X - \eta(X)\xi) = 0$. This holds for an arbitrary $X \in \chi(M)$; thus, we inferred the following:

$$\lambda = -\mu. \tag{3.15}$$

Invoking the relation 3.15 in 3.12, we easily obtained $\eta(\mathcal{L}_V \xi) = 0$. Because we have considered the potential vector field V as a contact vector field, there must exist a smooth function f such that $\mathcal{L}_V \xi = f\xi$. Making use of this in 3.12, we obtained $f = \lambda + \mu$. Therefore, by using the relation 3.15, we obtained $f = 0$ and thus $\mathcal{L}_V \xi = 0$. Finally, the Eq. 3.13 reduces to the following:

$$\mathcal{L}_V \eta = 0. \tag{3.16}$$

Thus, V is a strictly infinitesimal contact transformation.

Next, we considered the well-known formula given by Yano [35] as follows: $(\mathcal{L}_V \nabla)(X, Y) = \mathcal{L}_V \nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y$. Inserting $Y = \xi$ and using 2.7, $\mathcal{L}_V \xi = 0$ and 3.16 yields $(\mathcal{L}_V \nabla)(X, \xi) = 0$. Substituting this into 3.8, we deduced $QX = -2nX \forall X \in \chi(M)$, which settles our claim.

The *-η-Ricci soliton is a generalization of the *-Ricci soliton, where we considered $\mu = 0$ in 1.1 to obtain the *-Ricci soliton equation. We can rewrite the above theorem as follows:

Corollary 1. Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If the metric g represents a *-Ricci soliton and if the soliton vector field V is contact, then V is a strictly infinitesimal contact transformation and the manifold is Einstein.

Theorem 2. If the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfies the *-η-Ricci soliton (g, ξ, λ, μ) , where ξ is the Reeb vector field, then the soliton constants λ and μ are related by $\lambda = -\mu$.

Proof 2. Let M be a $(2n+1)$ -dimensional para-Kenmotsu manifold. Considering $V = \xi$ in the equation of a *-η-Ricci soliton 1.1 on M , we obtained the following:

$$(\mathcal{L}_\xi g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.17}$$

for all vector fields $X, Y, \in \chi(M)$. We combined the above equation with the identities 2.12, 3.3, 3.4 to yield the following:

$$[\lambda - 2n + 2]g(X, Y) - S(X, Y) + (\mu - 2)\eta(X)\eta(Y) = 0. \tag{3.18}$$

Now, we fed $Y = \xi$ into the previous equation and used the identity 2.9 to infer the following:

$$\lambda = -\mu, \tag{3.19}$$

as $\eta(X) \neq 0$, which finishes the proof.

Also, we see that if $\mu = 0$, then 3.19 gives $\lambda = 0$. Thus, we can state the following:

Corollary 2. If the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfies the *-Ricci soliton (g, ξ, λ) , where ξ is the Reeb vector field, then the soliton is steady.

Theorem 3. If the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfies the *-η-Ricci soliton (g, V, λ, μ) ,

where V is the gradient of a smooth function f , then the Laplacian equation satisfied by f is as follows:

$$\Delta(f) = (r + 4n^2) - \lambda(2n + 1) - \mu.$$

Proof 3. Now, we considered a *-η-Ricci soliton (g, V, λ, μ) on M as follows:

$$(\mathcal{L}_V g)(X, Y) + 2S^*(X, Y) + 2\lambda g(X, Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{3.20}$$

for all vector fields $X, Y, \in \chi(M)$.

We set $X = e_i$ and $Y = e_i$, in the above equation, where e_i denotes a local orthonormal frame; we then sum over $i = 1, 2, \dots, (2n + 1)$ and use 3.4 to obtain the following:

$$\text{div}V - (r + 4n^2) + \lambda(2n + 1) + \mu = 0. \tag{3.21}$$

If we assume that the vector field V is of gradient type, that is, $V = \text{grad}(f)$, where f is a smooth function on M , then the Eq. 3.21 provides the following:

$$\Delta(f) = (r + 4n^2) - \lambda(2n + 1) - \mu, \tag{3.22}$$

where $\Delta(f)$ is the Laplacian equation satisfied by f . This completes the proof.

If we replace the value of μ from 3.19 into the identity 3.21, λ takes the following form:

$$\lambda = \frac{(r + 4n^2) - \text{div}V}{2n}. \tag{3.23}$$

In view of 3.23, 3.19 becomes the following:

$$\mu = \frac{\text{div}V - (r + 4n^2)}{2n}. \tag{3.24}$$

Thus, we can state the following.

Corollary 3. If the metric of an $2n + 1$ -dimensional para-Kenmotsu manifold admits a *-η-Ricci soliton (g, V, λ, μ) , where V is the gradient of a smooth function f , the soliton constants λ and μ take the forms of $\lambda = \frac{(r+4n^2)-\text{div}V}{2n}$ and $\mu = \frac{\text{div}V-(r+4n^2)}{2n}$, respectively, where $\text{div}\xi$ is the divergence of the vector field ξ .

Theorem 4. Let the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfy the *-η-Ricci soliton (g, V, λ, μ) . Then the vector field V is solenoidal if $\lambda = \frac{r+4n^2}{2n}$ and $\mu = -\frac{(r+4n^2)}{2n}$.

Proof 4. consider the vector field V as solenoidal, that is, $\text{div}V = 0$; then, 3.23 and 3.24 provides the following:

$$\lambda = \frac{r + 4n^2}{2n}, \tag{3.25}$$

$$\mu = -\frac{r + 4n^2}{2n}. \tag{3.26}$$

Again, if we insert the value of λ and μ into the identity 3.21, we can obtain the following:

$$\text{div}V = 0, \tag{3.27}$$

that is, V is solenoidal, which ends our proof.

A vector field V is said to be a conformal Killing vector field if the following relation holds:

$$(\mathcal{L}_V g)(X, Y) = 2\Omega g(X, Y), \tag{3.28}$$

where Ω is some function of the coordinates (conformal scalar). Moreover, if Ω is not constant, the conformal Killing vector field V is said to be proper. In addition, when Ω is constant, V is called a homothetic vector field, and when the constant Ω becomes non-zero, V is said to be a proper homothetic vector field. If $\Omega = 0$ in the above equation, then V is called a Killing vector field.

Lemma 3. If the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfies the *-η-Ricci soliton (g, V, λ, μ) , where V is a conformal Killing vector field, then the manifold becomes η-Einstein.

Proof 5. Let (g, V, λ, μ) be a *-η-Ricci soliton on a $(2n+1)$ -dimensional para-Kenmotsu manifold M , where V is a conformal Killing vector field. Then from 1.1, 3.3, 3.4, 3.28, we can achieve the following:

$$S(X, Y) = [\lambda + \Omega - 2n + 1]g(X, Y) + (\mu - 1)\eta(X)\eta(Y), \tag{3.29}$$

which leads to the fact that the manifold is η-Einstein.

Lemma 4. Let the metric g of a $(2n+1)$ -dimensional para-Kenmotsu manifold satisfy the *-η-Ricci soliton (g, V, λ, μ) , where V is a conformal Killing vector field. Then V is one of the following cases:

- (1) proper vector field if $-\lambda + \mu$ is not constant;
- (2) homothetic vector field if $-\lambda + \mu$ is constant;
- (3) proper homothetic vector field if $-\lambda + \mu$ is a non-zero constant;
- (4) Killing vector field if $\lambda + \mu = 0$.

Proof 6. We combined $Y = \xi$ with identity 3.29 and used 2.9 and 3.4 to finally arrive at the following conclusion:

$$\Omega = -[\lambda + \mu], \tag{3.30}$$

because $\eta(X) \neq 0$. Now, using the properties of a conformal Killing vector field, we obtained our result.

4 GRADIENT ALMOST *-η-RICCI SOLITON ON PARA-KENMOTSU MANIFOLDS

In this section, we have studied gradient almost *-η-Ricci solitons on para-Kenmotsu manifolds.

Theorem 5. Let $M^{(2n+1)}(\phi, \xi, \eta, g)$ be a para-Kenmotsu manifold. If the metric g represents a gradient almost *-η-Ricci soliton, then either M is Einstein or there exists an open set where the potential vector field V is pointwise collinear with the characteristic vector field ξ .

Proof 7. In view of 3.3, in the definition of the gradient almost *-η-Ricci soliton given by Eq. 1.2, we acquired the following:

$$\nabla_X Df = QX - (\lambda - 2n + 1)X - (\mu - 1)\eta(X)\xi,$$

for any vector field X on M. We took the covariant derivative in the previous equation along an arbitrary vector Y and used 2.7 and 2.8 to yield the following:

$$\begin{aligned} \nabla_Y \nabla_X Df &= (\nabla_Y Q)X + Q(\nabla_Y X) - Y(\lambda)X - (\lambda + 2n - 1)(\nabla_Y X) \\ &\quad - (\mu - 1)\{g(X, Y)\xi - 2\eta(X)\eta(Y)\xi \\ &\quad + \eta(\nabla_Y X)\xi + \eta(X)Y\}. \end{aligned}$$

Applying this in the expression of the Riemannian curvature tensor, we obtained the following:

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + Y(\lambda)X - X(\lambda)Y - (\mu - 1)\{\eta(Y)X - \eta(X)Y\}. \tag{4.1}$$

Moreover, considering an inner product w.r.t. ξ and using 3.1, 3.2 yields the following:

$$g(R(X, Y)Df, \xi) = Y(\lambda)\eta(X) - X(\lambda)\eta(Y), \tag{4.2}$$

for X, Y ∈ χ(M). Furthermore, the inner product of 2.10 with the potential vector field Df provides the following:

$$g(R(X, Y)Df, \xi) = \eta(Y)X(f) - \eta(X)Y(f), \tag{4.3}$$

for arbitrary X and Y on M. Comparing 4.2 and 4.3 and plugging Y = ξ, we have X(f + λ) = ξ(f + λ)η(X). From this, we achieved the following:

$$d(f + \lambda) = \xi(f + \lambda)\eta. \tag{4.4}$$

So, if (f + λ) is invariant along the distribution Ker(η), that is, if X ∈ Ker(η), then X(f + λ) = d(f + λ)X = 0.

Now, if we take the inner product w.r.t. an arbitrary vector field Z after plugging X = ξ into 4.1, we obtained the following:

$$g(R(\xi, Y)Df, Z) = S(Y, Z) + (2n - \xi(\lambda) + \mu - 1)g(Y, Z) + Y(\lambda)\eta(Z) - (\mu - 1)\eta(Y)\eta(Z). \tag{4.5}$$

Again, from 2.10, we can easily deduce the following for arbitrary vector fields Y and Z on M:

$$g(R(\xi, Y)Df, Z) = \xi(f)g(Y, Z) - Y(f)\eta(Z). \tag{4.6}$$

Comparing Eqs. 4.5, 4.6 and applying 4.4, we obtained the following:

$$\begin{aligned} S(Y, Z) &= \{\xi(f + \lambda) - \mu - 2n - 1\}g(Y, Z) \\ &\quad - \{\xi(f + \lambda) - \mu - 1\}\eta(Y)\eta(Z). \end{aligned} \tag{4.7}$$

As the above equation holds true for arbitrary Y and Z, the manifold is η-Einstein. Now, contracting 4.7, we inferred the following:

$$\xi(f + \lambda) = \frac{r}{2n} + \mu + 2n + 2. \tag{4.8}$$

Plugging this into 4.7, we acquired the following:

$$S(Y, Z) = \left(\frac{r}{2n} + 1\right)g(Y, Z) - \left(\frac{r}{2n} + 2n + 1\right)\eta(Y)\eta(Z),$$

for arbitrary vector fields Y and Z on M, which is exactly the same as 2.14. Now, contracting 4.1 w.r.t. X reduces to the following:

$$S(Y, Df) = \frac{1}{2}Y(r) + 2nY(\lambda) - 2n(\mu + 1)\eta(Y), \tag{4.9}$$

which holds for any Y ∈ χ(M). Now, taking into with 2.14, we computed the following:

$$\begin{aligned} (r + 2n)Y(f) - (r + 2n(2n + 1))\eta(Y)\xi(f) - nY(r) \\ - 4n^2Y(\lambda) + 4n^2(\mu + 1)\eta(Y) = 0, \end{aligned} \tag{4.10}$$

for all Y ∈ χ(M). Now, setting Y = ξ and then in view of 4.8, we easily derive the following relation:

$$\xi(r) = -2(r + 2n(2n + 1)). \tag{4.11}$$

Because d² = 0 and dη = 0, from 4.4, it follows that dr ∧ η = 0, that is, dr(X)η(Y) - dr(Y)η(X) = 0 for arbitrary X, Y ∈ χ(M). After inserting Y = ξ and applying 4.11, it reduces to X(r) = -2(r + 2n(2n + 1))ξ. As X is an arbitrary vector field, we conclude the following:

$$Dr = -2(r + 2n(2n + 1))\xi.$$

Let X be a vector field of the distribution Ker(η). Then, 4.10 provides the following:

$$(r + 2n)X(f) - 4n^2X(\lambda) = 0.$$

Invoking 4.4 and 4.8 we obtained (r + 2n(2n + 1))X(f) = 0. From here we concluded the following:

$$(r + 2n(2n + 1))(Df - \xi(f)\xi) = 0.$$

If r = -2n(2n + 1), then from 2.14 we can determine that the manifold is Einstein with Einstein constant -2n.

If r ≠ -2n(2n + 1) on some open set O of M, then Df = ξ(f)ξ on that open set, that is, the potential vector field is pointwise collinear with the characteristic vector field ξ, which finishes the proof.

Corollary 4. Let (g, V, λ, μ) be a gradient *-η-Ricci soliton on a (2n+1)-dimensional para-kenmotsu manifold. Then the potential vector field V is pointwise collinear with the characteristic vector field ξ.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

Conceptualization and methodology: SD and NT; formal analysis: SD and NT; writing—original draft preparation: SD;

writing–review and editing: SD and NT; supervision: SD and NT; project administration: SD and NT; and funding acquisition: SD and NT. All authors have read and agreed to the published version of the manuscript.

REFERENCES

1. Topping P. *Lecture on the Ricci Flow*. London: Cambridge University Press (2006). doi:10.1017/CBO9780511721465
2. Perelman G. *The Entropy Formula for the Ricci Flow and its Geometric Applications*. Preprint. Available from: <http://arXiv.org/abs/math.DG/0211159> (Accessed November 11, 2002).
3. Cho JT, Kimura M. Ricci Solitons and Real Hypersurfaces in a Complex Space Form. *Tohoku Math J* (2009) 61(2):205–12. Second Series. doi:10.2748/tmj/1245849443
4. Kaimakamis G, Panagiotidou K. * η -Ricci Solitons of Real Hypersurfaces in Non-flat Complex Space Forms. *J Geometry Phys* (2014) 86:408–13. doi:10.1016/j.geomphys.2014.09.004
5. Tachibana S. On Almost-Analytic Vectors in Almost Kählerian Manifolds. *Tohoku Math.J* (1959) 11(2):247–65. doi:10.2748/tmj/1178244584
6. Dey S, Roy S. * η -Ricci Soliton within the Framework of Sasakian Manifold. *J Dynamical Syst Geometric Theories* (2020) 18(2):163–81. doi:10.1080/1726037x.2020.1856339
7. Bejan CL, Crasmareanu M. Second Order Parallel Tensors and Ricci Solitons in 3-dimensional normal Paracontact Geometry. *Ann Glob Anal Geom* (2014) 46(2):117–27. doi:10.1007/s10455-014-9414-4
8. Catino G, Mazzieri L. Gradient Einstein Solitons. *Nonlinear Anal* (2016) 132:66–94. doi:10.1016/j.na.2015.10.021
9. Calvaruso G, Perrone D. Ricci Solitons in Three-Dimensional Paracontact Geometry. *J Geom Phys* (2015) 98:1–12. doi:10.1016/j.geomphys.2015.07.021
10. Prakasha DG, Hadimani BS. η -Ricci Solitons on Para-Sasakian Manifolds. *J Geom* (2017) 108:383–92. doi:10.1007/s00022-016-0345-z
11. Ghosh A, Patra DS. * η -Ricci Soliton within the Framework of Sasakian and (κ, μ) -contact Manifold. *Int J Geom Methods Mod Phys* (2018) 15(7):1850120. doi:10.1142/s0219887818501207
12. Roy S, Dey S, Bhattacharyya A, Hui SK. * η -Conformal η -Ricci Soliton on Sasakian Manifold. *Asian-European J Math* (2021)(17) 2250035. doi:10.1142/S1793557122500358
13. Roy S, Dey S, Bhattacharyya A. Conformal Einstein Soliton within the Framework of Para-Kähler Manifold. arXiv:2005.05616v1 [math.DG]. *Differential Geometry-Dynamical Syst* (2021) 23:235–43.
14. Roy S, Dey S, Bhattacharyya A. Yamabe Solitons on $(LCS)_n$ -Manifolds. *J Dynamical Syst Geometric Theories* (2020) 18(2):261–79. doi:10.1080/1726037x.2020.1868100
15. Roy S, Dey S, Bhattacharyya A. Some Results on η -Yamabe Solitons in 3-dimensional Trans-sasakian Manifold. accepted for publication in Carpathian Mathematical Publications (2021). arXiv:2001.09271v2 [math.DG].
16. Roy S, Dey S, Bhattacharyya A. A Kenmotsu Metric as a Conformal η -Einstein Soliton. *Carpathian Math Publ* (2021) 13(1):110–8. doi:10.15330/cmp.13.1.110-118
17. Roy S, Dey S, Bhattacharyya A. Conformal Yamabe Soliton and * η -Yamabe Soliton with Torse Forming Potential Vector Field. *Matematika* (2021) 73(4):282–92.
18. Ganguly D, Dey S, Ali A, Bhattacharyya A. Conformal Ricci Soliton and Quasi-Yamabe Soliton on Generalized Sasakian Space Form. *J Geometry Phys* (2021) 169:104339. doi:10.1016/j.geomphys.2021.104339
19. Hamada T. Real Hypersurfaces of Complex Space Forms in Terms of Ricci * η -tensor. *Tokyo J.Math.* (2002) 25:473–83. doi:10.3836/tjm/1244208866
20. Dey S, Sarkar S, Bhattacharyya A. * η -Ricci Soliton and Contact Geometry. accepted for publication in Ricerche di Matematica. Italy: Springer-Verlag (2021). doi:10.1007/s11587-021-00667-0

FUNDING

This project was supported by the Researchers Supporting Project number (RSP2022R413), King Saud University, Riyadh, Saudi Arabia.

21. Sarkar S, Dey S, Bhattacharyya A. Ricci Solitons and Certain Related Metrics on 3-dimensional Trans-Sasakian Manifold. arXiv:2106.10722v1 [math.DG] (2021).
22. Prakasha DG, Veerasha P. Para-Sasakian Manifolds and * η -Ricci Solitons. *Afr Mat* (2019) 30:989–98. doi:10.1007/s13370-019-00698-9
23. Venkatesha V, Kumara HA, Naik DM. Almost * η -Ricci Soliton on ParaKenmotsu Manifolds. *Arab J Math* (2019) 9:715–26. doi:10.1007/s40065-019-00269-7
24. Küpeli Erken I. Yamabe Solitons on Three-Dimensional Normal Almost Paracontact Metric Manifolds. *Period Math Hung* (2020) 80(2):172–84. doi:10.1007/s10998-019-00303-3
25. Sarkar S, Dey S, Chen X. Certain Results of Conformal and * η -Conformal Ricci Soliton on Para-Cosymplectic and Para-Kenmotsu Manifolds. accepted for publication in Filomat. *Filomat* (2021) 35(15):5001–5015. doi:10.2298/FIL2115001S
26. Deshmukh S, Alsodais H, Bin Turki N. Some Results on Ricci Almost Solitons. *Symmetry* (2021) 13:430. doi:10.3390/sym13030430
27. Sarkar S, Dey S, Bhattacharyya A. Geometry of Para-Sasakian Metric as an Almost Conformal η -Ricci Soliton. arXiv:2109.05448v1 [math.DG] (2021).
28. Naik DM, Venkatesha V. η -Ricci Soliton and Almost η -Ricci Soliton on Para-Sasakian Manifolds. *Int J Geom Methods Mod Phys* (2019) 16(9):1950134. doi:10.1142/s0219887819501342
29. Welyczko J. Slant Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds. *Mediterr J Math* (2014) 11(3):965–78. doi:10.1007/s00009-013-0361-2
30. Kenmotsu K. A Class of Almost Contact Riemannian Manifolds. *Tohoku Math J* (1972) 24:93–103. doi:10.2748/tmj/1178241594
31. Zamkovoy S. On Para-Kenmotsu Manifolds. *Filomat* (2018) 32(14). doi:10.2298/fil1814971z
32. Blaga AM. η -Ricci Solitons on Para-Kenmotsu Manifolds. *Balkan J Geom Appl* (2015) 20(1):1–13.
33. Blaga AM. Almost η -Ricci Solitons in $(LCS)_n$ -manifolds. *Bull Belgian Math Soc Simon Stevin* (2018) 25(5):641–53. doi:10.36045/bbms/1547780426
34. Blaga AM, Ozgur C. Almost η -Ricci and Almost η -Yamabe Solitons with Torse-Forming Potential Vector Field. arXiv preprint arXiv:2003.12574 (2020).
35. Yano K. *Integral Formulas in Riemannian Geometry, Pure and Applied Mathematics*. New York: Marcel Dekker (1970).

Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's Note: All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors, and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

Copyright © 2022 Dey and Turki. This is an open-access article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) and the copyright owner(s) are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.