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Normal Curves in 4-Dimensional Galilean Space G^4

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In this article, first, we give the definition of normal curves in 4-dimensional Galilean space G^4. Second, we state the necessary condition for a curve of curvatures $\tau(s)$ and $\sigma(s)$ to be a normal curve in 4-dimensional Galilean space G^4. Finally, we give some characterizations of normal curves with constant curvatures in G^4.

Keywords: normal curves, Galilean space, curvatures, W-curve, Frenet apparatus

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1. INTRODUCTION

Galilean geometry is one of the Cayley-Klein geometries whose motions are the Galilean transformations of classical kinematics [1]. The Galilean transformation group has an important place in classical and modern physics. It is well known that the idea of world lines originates in physics and was pioneered by Einstein. The world line of a particle is just the curve in space-time which indicates its trajectory [2].

In Euclidean 3-space E^3, there are three types of curves, namely, osculating, rectifying, and normal curves [3]. The osculating curve in E^3 is defined as a curve whose position vector always lies in its osculating plane, which is spanned by the tangent vector T and the normal vector N [3]. The rectifying curve in E^3 is defined as a curve whose position vector always lies in its rectifying plane, which is spanned by the tangent vector T and the binormal vector B. Many researchers have investigated rectifying curves in Euclidean, Lorentz-Minkowski, and Galilean space, as can be seen in [4–6]. Similarly, a normal curve in E^3 is defined as a curve whose position vector always lies in its normal plane, which is spanned by the normal vector N and the binormal vector B of the curve. Normal curves in n-dimensional Euclidean space was studied by Ozcan Bekats [7], framed normal curves in Euclidean space was studied by B.D. Yazici, S. O.Karakus, and M. Tosun [8], and normal curves on a smooth immersed surface was investigated by A.A. Shaikh, M.S. Lone, and P.R. Ghosh [9].

There are also many studies related to normal curves in non-Euclidean spaces, for example, normal curves and their characterizations in Lorentzian n-space was studied by Ozgür Boyacıoğlu Kalkan [10] and to the classification of normal and osculating curves in 3-dimensional Sasakian space was studied by M. Kulahck, M. Bahatas, and A. Bilici [11].

In recent years, researchers have begun to introduce curves and surfaces in Galilean and pesedo-Galilean spaces [12–24]. Normal and rectifying curves in Galilean G^3 were obtained by Handan Oztekin [25]. Also, many studies about Galilean Geometry were found in Reference [1]. Frenet-Serret frame in the Galilean 4-space was constructed by S.Yilmaz [26].

In the present study, we considered a curve in Galilean 4-space G^4 whose position vector satisfies the equation $\alpha(s) = \lambda(s)N(s) + \mu_1(s)B_1(s) + \mu_2(s)B_2(s)$ for differentiable functions $\lambda(s)$, $\mu_1(s)$, and $\mu_2(s)$. $N(s)$, $B_1(s)$, and $B_2(s)$ are normal, first binormal, and second binormal vectors of the curve in Galilean space G^4. In the first part of the study, the necessary condition for a curve to be a normal curve was obtained; then, we considered a special case when the curvatures are constant and got the position vector of the normal curve in G^4. At the end of the study, it can be seen that the normal curve in G^4 lies on a sphere if $\frac{\tau}{\sigma}$ = constant, where τ and σ are the second and the third curvatures of the normal curve $\alpha(s)$.

2. PRELIMINARIES

In this section, we will give some definitions considered in this study. Let $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ and $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ be two vectors in G^4. The Galilean scalar product in G^4 is defined by

$$< \overrightarrow{x}, \overrightarrow{y} >_{G^4} = \begin{cases} x_1 y_1, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2y_2 + x_3y_3 + x_4y_4, & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

The norm of the vector $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ is given by $|\overrightarrow{x}| = \sqrt{< \overrightarrow{x}, \overrightarrow{x} >_{G^4}}$.

The cross product of any three vectors $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{z}$ in G^4 is defined by the relation

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0 \text{ or } z_1 \neq 0, \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, & \text{if } x_1 = y_1 = z_1 = 0, \end{cases}$$

where the unit vectors $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0)$, and $e_4 = (0, 0, 0, 1)$ [1].

A curve $\alpha : I \subset R \rightarrow G^4$ of C^∞ in the Galilean space G^4 is defined by $\alpha(t) = (x(t), y(t), z(t), w(t))$.

If the curve is parameterized by Galilean arc-length s, it is defined by $\alpha(s) = (s, y(s), z(s), w(s))$.

The Frenet frame for the parameterized curve $\alpha(s) = (s, y(s), z(s), w(s))$ in G^4 is denoted by the following vectors

$T(s) = \alpha'(s) = (1, y'(s), z'(s), w'(s))$,

$N(s) = \frac{1}{k(s)}\alpha''(s) = \frac{1}{k(s)}(0, y''(s), z''(s), w''(s))$,

$B_1(s) = \frac{1}{\tau(s)}(0, (\frac{y''(s)}{k(s)})', (\frac{z''(s)}{k(s)})', (\frac{w''(s)}{k(s)})')$,

$B_2(s) = \varsigma T(s) \times N(s) \times B_1(s)$,

Here, the coefficient ς is taken ± 1 to make the determinant of the matrix $[T, N, B_1, B_2] = 1$.

where $T(s), N(s), B_1(s)$ and $B_2(s)$ are the tangent, normal, the first binormal, and the second binormal vectors of $\alpha(s)$. $k(s)$ and $\tau(s)$ are the first and second curvatures, which are given by the following equations

$k(s) = \left|T'(s)\right|_{G^4} = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2}$,

$\tau(s) = \left|N'(s)\right|_{G^4} = \sqrt{< N'(s), N'(s) >_{G^4}}$.

The third curvature of the parameterized curve $\alpha(s)$ is denoted by $\sigma(s) =< B_1'(s), B_2(s) >_{G^4}$. If the curvatures of $\alpha(s)$ are constants, the curve $\alpha(s)$ is called a W-curve. The set $\{T(s), N(s), B_1(s), B_2(s), k(s), \tau(s), \sigma(s)\}$ is called Frenet apparatus of the curve $\alpha(s)$. The vectors $T(s), N(s), B_1(s)$, and $B_2(s)$ are mutually orthogonal.

$< T(s), T(s) >_{G^4} =< N(s), N(s) >_{G^4} =< B_1(s), B_1(s) >_{G^4}$
$=< B_2(s), B_2(s) >_{G^4} = 1$,
and $< T(s), N(s) >_{G^4} =< T(s), B_1(s) >_{G^4} =< T(s), B_2(s) >_{G^4} =< N(s), B_1(s) >_{G^4}$
$=< N(s), B_2(s) >_{G^4} =< B_1(s), B_2(s) >_{G^4} = 0$.

The derivatives of the Frenet equations are defined by [26].

$$\begin{aligned} T'(s) &= k(s)N(s), \\ N'(s) &= \tau(s)B_1(s), \\ B_1'(s) &= -\tau(s)N(s) + \sigma(s)B_2(s), \\ B_2'(s) &= -\sigma(s)B_1(s). \end{aligned} \tag{2.1}$$

3. NORMAL CURVES IN G^4

In the following section, we will define the normal curves in Galilean 4-dimensional space and prove that there are no congruent curves to the normal curve $\alpha(s)$; finally, we will provide some characterizations of the normal curves in G^4.

Definition 1. *Let $\alpha : I \subset R \rightarrow G^4$ be a parameterized curve in G^4. A curve $\alpha(s)$ is called a normal curve if the orthogonal components of $T(s)$ contains a fixed point for all $s \in I$.*

In the following theorem, we indicate the position vector of the normal curve in Galilean 4-space G^4.

Theorem 1. *The position vector of the normal curve in G^4 with curvatures $k(s)$, $\tau(s)$, and $\sigma(s)$ are defined if $\tau(s)$ and $\sigma(s)$ satisfy the following equations:*

$$\Big(\mu_1(s)\Big)^{(r+2)} + \sum_{k=0}^{r}\sum_{\ell=0}^{r-k}\binom{r+1}{k}\binom{r-k}{\ell}\Big[\Big(\sigma(s)\Big)^{(k)}\Big(\sigma(s)\Big)^{(\ell)} + \Big(\tau(s)\Big)^{(k)}\Big(\tau(s)\Big)^{(\ell)}\Big]\Big(\mu_1(s)\Big)^{(r-k-\ell)} = 0.$$

Proof: The position vector of a normal curve in G^4 is defined by

$$\alpha(s) = \lambda(s)N(s) + \mu_1(s)B_1(s) + \mu_2(s)B_2(s), \tag{3.1}$$

where $\lambda(s)$, $\mu_1(s)$, and $\mu_2(s)$ are smooth functions of s.

By differentiating (Equation 3.1) with respect to s, we obtain

$$\begin{aligned} \alpha'(s) &= \lambda'(s)N(s) + \lambda(s)N'(s) + \mu_1'(s)B_1(s) + \mu_1(s)B_1'(s) \\ &+ \mu_2'(s)B_2(s) + \mu_2(s)B_2'(s). \end{aligned} \tag{3.2}$$

substituting Frenet (Equations 2.1) into Equation 3.2, we have $T(s) = (\lambda'(s) - \tau(s)\mu_1(s))N(s) + (\lambda(s)\tau(s) + \mu_1'(s) - \mu_2(s)\sigma(s))B_1(s) + (\mu_1(s)\sigma(s) + \mu_2'(s))B_2(s)$.

Hence, we have the following system of differential equations:

$$\frac{d\lambda}{ds} = \tau(s)\mu_1(s), \tag{3.3}$$

$$\frac{d\mu_1}{ds} = -\tau(s)\lambda(s) + \sigma(s)\mu_2(s), \tag{3.4}$$

and

$$\frac{d\mu_2}{ds} = -\sigma(s)\mu_1(s). \tag{3.5}$$

Since the curvature functions $\tau(s)$ and $\sigma(s)$ must be differentiable functions, so we consider a set $\mathcal{F}$ of differentiable functions, defined by

$$\mathcal{F} = \{P_n(s),\ e^{\omega s}, \cos\beta s, \sin\beta s,\ n \in \mathbb{N},\ \omega,\ \beta \in \mathbb{R}\}, \tag{3.6}$$

where $P_n(s)$ denotes a polynomial function of degree n in s. Here, the curvature functions can be any function of $\mathcal{F}$, a linear combination or a product of these functions. It is said that $\mathcal{L}[D_s]$ annihilates the function $\Upsilon(s)$ if $\mathcal{L}[D_s](\Upsilon(s)) = 0$. Here, all the member functions of $\mathcal{F}$ have this property. Consequently, we have two linear differential operators, $\Phi(D_s)$ and $\Psi(D_s)$, such that $\Phi(D_s)\{\tau(s)\} = 0$ and $\Psi(D_s)\{\sigma(s)\} = 0$, where $D_s = \frac{d}{ds}$, some of these annihilator operators are listed in **Table 1**. By applying the operator $\Theta(D_s) = \Phi(D_s)\Psi(D_s)$ on Equation (3.4), we have

$$\begin{aligned}\Theta(D_s)D_s\mu_1(s) &= -\Theta(D_s)\{\tau(s)\lambda(s)\} + \Theta(D_s)\{\sigma(s)\mu_2(s)\}\\ &= -\Psi(D_s)\Phi(D_s)\{\tau(s)\lambda(s)\}\\ &\quad+ \Phi(D_s)\Psi(D_s)\{\sigma(s)\mu_2(s)\}.\end{aligned} \tag{3.7}$$

The operator $\Theta(D_s)$ annihilates the terms that contain $\lambda(s)$ and $\mu_2(s)$, i.e., the right hand side (RHS) of Equation (3.7) contains the derivatives of $\lambda(s)$ and $\mu_2(s)$ from order 1 to an order of less than the order of the derivative of $\mu_1(s)$ by 1 at most. Let the maximum derivative in the left hand side (LHS) of Equation (3.7) be $r+1$; then, its RHS can be written as a linear combination of the derivatives of $\tau^{(k)}(s)\lambda^{(r-k)}(s)$ and $\sigma^{(k)}(s)\mu_2^{(r-k)}(s)$ and $k \in \{0, 1, \ldots, r-1\}$, consequently, we have

$$\begin{aligned}\Theta(D_s)D_s\mu_1(s) &= \sum_{k=0}^{r-1}\Big(a_{r,k}\big(\sigma(s)\big)^{(k)}\big(\mu_2(s)\big)^{(r-k)}\\ &\quad- b_{r,k}\big(\tau(s)\big)^{(k)}\big(\lambda(s)\big)^{(r-k)}\Big),\end{aligned} \tag{3.8}$$

where $a_{r,k},\ b_{r,k} \in \mathbb{R}$. Next by differentiating Equations (3.3) and (3.5), $(r-1-k)$ times and applying Leibniz's rule, one gets

$$\begin{aligned}\big(\lambda(s)\big)^{(r-k)} &= \big(\tau(s)\mu_1(s)\big)^{(r-1-k)}\\ &= \sum_{\ell=0}^{r-1-k}\binom{r-1-k}{\ell}\big(\tau(s)\big)^{(\ell)}\big(\mu_1(s)\big)^{(r-1-k-\ell)},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\big(\mu_2(s)\big)^{(r-k)} &= -\big(\sigma(s)\mu_1(s)\big)^{(r-1-k)}\\ &= -\sum_{\ell=0}^{r-1-k}\binom{r-1-k}{\ell}\big(\sigma(s)\big)^{(\ell)}\big(\mu_1(s)\big)^{(r-1-k-\ell)}\end{aligned} \tag{3.10}$$

Applying Equations (3.9) and (3.10) into (3.8) yields

TABLE 1 | Annihilator operators of some functions.

Function	The corresponding annihilator operator
$P_n(s)$	$D_s^{n+1}\{P_n(s)\} = 0$
$e^{\omega s}$	$(D_s - \omega)\{e^{\omega s}\} = 0$
$\cos\beta s \setminus \sin\beta s$	$(D_s^2 + \beta^2)\{\cos\beta s \setminus \sin\beta s\} = 0$
$s^n e^{\omega s}$	$(D_s - \omega)^{n+1}\{s^n e^{\omega s}\} = 0$
$s^n e^{\omega s}\cos\beta s$	$\left(D_s^2 - 2\omega D_s + \omega^2 + \beta^2\right)^{n+1}\{s^n e^{\omega s}\cos\beta s\} = 0$

$$\begin{aligned}\Theta(D_s)D_s\mu_1(s)+\sum_{k=0}^{r-1}\sum_{\ell=0}^{r-1-k}\binom{r-1-k}{\ell}\Big[a_{r,k}\big(\sigma(s)\big)^{(k)}\big(\sigma(s)\big)^{(\ell)}+\\ b_{r,k}\big(\tau(s)\big)^{(k)}\big(\tau(s)\big)^{(\ell)}\Big]\big(\mu_1(s)\big)^{(r-1-k-\ell)} = 0.\end{aligned} \tag{3.11}$$

Equation (3.11) is a linear differential equation of order $(r+1)$ in $\mu_1(s)$ with differential variable coefficients. Its general solution depends on the nature of the coefficients of its derivatives. However, the power series method can be applied to obtain its solution, especially, for any value of $s \in \mathbb{R}$ is an ordinary point in all the coefficients of the derivatives $\mu_1^{(k)}(s)$, $k = 0, 1, \ldots, r$.
Next, we studied the case when $\tau(s) \in P_n(s)$ and $\sigma(s) \in P_m(s)$. When $r = \max(m, n)$, then the annihilator operator is D_s^{r+1}, $D_s^{r+1}\tau(s) = D_s^{r+1}\sigma(s) = 0$, and consequently, $\Theta(D_s) = D_s^{r+1}$ was applied into Equation (3.7) by applying Leibniz's rule, yielding

$$\begin{aligned}\big(\mu_1(s)\big)^{(r+2)} &= \sum_{k=0}^{r}\binom{r+1}{k}\Big[\big(\sigma(s)\big)^{(k)}\big(\mu_2(s)\big)^{(r+1-k)}\\ &\quad- \big(\tau(s)\big)^{(k)}\big(\lambda(s)\big)^{(r+1-k)}\Big].\end{aligned} \tag{3.12}$$

By using Equations (3.9) and (3.10), Equation (3.12) becomes

$$\begin{aligned}\big(\mu_1(s)\big)^{(r+2)}+\sum_{k=0}^{r}\sum_{\ell=0}^{r-k}\binom{r+1}{k}\binom{r-k}{\ell}\Big[\big(\sigma(s)\big)^{(k)}\big(\sigma(s)\big)^{(\ell)}+\\ \big(\tau(s)\big)^{(k)}\big(\tau(s)\big)^{(\ell)}\Big]\big(\mu_1(s)\big)^{(r-k-\ell)} = 0.\end{aligned} \tag{3.13}$$

Corollary 1. *Let $\alpha(s)$ be a normal curve in G^4. If the curvatures $\tau(s)$ and $\sigma(s) \in P_1(s)$, then the position vector of the normal curve in G^4 is given by*

$$\begin{aligned}\alpha(s) &= \left(\frac{\pi}{4}\sqrt{2}\left(\left(C\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2 + \left(S\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2\right) + c_\lambda\right)\\ &N(s) + \left(C_1\sin\left(\frac{\sqrt{2}}{2}(s+1)^2\right) + C_2\cos\left(\frac{\sqrt{2}}{2}(s+1)^2\right)\right.\\ &\left.+C_3\left({}_1F_2\left[\begin{matrix}1\\ \frac{3}{4}\ \frac{5}{4}\end{matrix};-\frac{1}{8}(s+1)^4\right]\right)\right)\\ &B_1(s) \quad + \quad \left(-\frac{\pi}{4}\sqrt{2}\left(\left(C\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2 + \left(S\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2\right)\right.\\ &\left.+c_{\mu_2}\right) B_2(s)\end{aligned}$$

where C_1, C_2, C_3, c_λ, and c_{μ_2} are constants; the Fresnel functions, $C(s) = \int_0^s \cos(t^2)dt$; and $S(s) = \int_0^s \sin(t^2)dt$; the generalized hypergeometric function,

$$ {}_pF_q\left[\begin{matrix}\beta_1 & \beta_2 & \dots & \beta_p\\ \gamma_1 & \gamma_2 & \dots & \gamma_q\end{matrix}; s\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\beta_p)_n s^n}{\prod_{j=1}^{p}(\gamma_q)_n n!}; $$

and $(\beta)_n$ is the shifted factorial, defined by [27]

$$ (\beta)_n = \beta(\beta+1)\dots(\beta+n-1) = \prod_{j=0}^{n-1}(\beta+j), \text{ and } (\beta)_0 = 1. $$

Proof: We consider a special case, when $\tau(s), \sigma(s) \in P_1(s)$, i.e., $\tau(s) = a_1 s + b_1$, $\sigma(s) = a_2 s + b_2$, then we have $r = 1$, $\tau'(s) = a_1$ and $\sigma'(s) = a_2$ and the annihilate operator is D_s^2 ($D_s^2\tau(s) = \tau''(s) = D_s^2\sigma(s) = 0$); by substituting into Equation (3.13), one gets

$$ \left(\mu_1(s)\right)^{(3)} + \sum_{k=0}^{1}\sum_{\ell=0}^{1-k}\binom{2}{k}\binom{1-k}{\ell}\left[\left(\sigma(s)\right)^{(k)}\left(\sigma(s)\right)^{(\ell)} + \left(\tau(s)\right)^{(k)}\left(\tau(s)\right)^{(\ell)}\right]\left(\mu_1(s)\right)^{(1-k-\ell)} = 0. \tag{3.14} $$

By extracting the summations, we have

$$ \frac{d^3\mu_1}{ds^3} + \left(\left(\sigma(s)\right)^2 + \left(\tau(s)\right)^2\right)\frac{d\mu_1}{ds} + 3(\sigma(s)\sigma'(s) + \tau(s)\tau'(s))\mu_1(s) = 0. \tag{3.15} $$

By applying the power series method to solve Equation (3.15), where $\mu_1(s) = \sum_{k=0}^{\infty} c_k s^k$, differentiating it for up to three times, then substituting into Equation (3.15), and collecting the coefficients, one gets

$$ \begin{aligned} c_3 &= \tfrac{-1}{3!}\sum_{j=0}^{1}(j+1)\varpi_{1-j}c_j,\\ c_4 &= \tfrac{1}{2(4!)}\sum_{j=0}^{2}(5j^2-9j-3)\varpi_{2-j}c_j,\\ c_{n+3} &= \tfrac{1}{2(n+1)_3}\sum_{j=0}^{2}\left((5n+3)j^2-(7n+11)j-8(n-1)\right)\\ &\quad \varpi_{2-j}c_{n-1+j},\ n = 2,3,\dots, \end{aligned} $$

where,

$$ \varpi_0 = b_1^2 + b_2^2,\ \ \varpi_1 = a_1b_1 + a_2b_2,\ \ \varpi_2 = a_1^2 + a_2^2. $$

Finally, for a special case, when $a_1 = a_2 = b_1 = b_2 = 1$, the resultant differential equation is

$$ \mu_1''' + 2(s+1)^2\mu_1' + 6(s+1)\mu_1 = 0, \tag{3.16} $$

which has the solution
$\mu_1(s) = C_1\sin\left(\frac{\sqrt{2}}{2}(s+1)^2\right) + C_2\cos\left(\frac{\sqrt{2}}{2}(s+1)^2\right) + C_3\left({}_1F_2\left[\begin{matrix}1\\ \frac{3}{4}\ \frac{5}{4}\end{matrix}; -\frac{1}{8}(s+1)^4\right]\right)$.
It is worth noting that the generalized hypergeometric function is convergent, when $p < q+1$, which holds in the proposed problem. Hence, we can obtain the functions $\lambda(s)$ and $\mu_2(s)$, by integrating Equations (3.3) and (3.5) with respect to s, respectively, then we have

$$ \lambda(s) = \frac{\pi}{4}\sqrt{2}\left(\left(C\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2 + \left(S\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2\right) + c_\lambda, $$

and

$$ \mu_2(s) = -\frac{\pi}{4}\sqrt{2}\left(\left(C\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2 + \left(S\left(\frac{\sqrt[4]{2}(s+1)}{\sqrt{\pi}}\right)\right)^2\right) + c_{\mu_2}. $$

Corollary 2. *The position vector of the normal curve in G^4 with constant curvatures τ and σ is given by*

$$ \begin{aligned} \alpha(s) = {} & \left[\tfrac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1\sin\sqrt{\tau^2+\sigma^2}s - c_2\cos\sqrt{\tau^2+\sigma^2}s) + c_3\right]\\ & N(s) + (c_1\cos\sqrt{\tau^2+\sigma^2}s + c_2\sin\sqrt{\tau^2+\sigma^2}s)B_1(s) +\\ & \left[\tfrac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1\sin\sqrt{\tau^2+\sigma^2}s - c_2\cos\sqrt{\tau^2+\sigma^2}s) + c_4\right]B_2(s). \end{aligned} $$

Proof: The position vector of a normal curve in G^4 is defined by

$$ \alpha(s) = \lambda(s)N(s) + \mu_1(s)B_1(s) + \mu_2(s)B_2(s), \tag{3.17} $$

where $\lambda(s)$, $\mu_1(s)$, and $\mu_2(s)$ are differentiable functions of s.

By differentiating Equation (3.17) with respect to s, we obtain

$$ \alpha'(s) = \lambda'(s)N(s) + \lambda(s)N'(s) + \mu_1'(s)B_1(s) + \mu_1(s)B_1'(s) + \mu_2'(s)B_2(s) + \mu_2(s)B_2'(s). \tag{3.18} $$

Substituting Frenet Equations (2.1) into Equation (3.18), we have
$T(s) = (\lambda'(s) - \tau(s)\mu_1(s))N(s) + (\lambda(s)\tau(s) + \mu_1'(s) - \mu_2(s)\sigma(s))B_1(s) + (\mu_1(s)\sigma(s) + \mu_2'(s))B_2(s)$.

Hence, we obtain the following differential equations:

$$ \begin{aligned} \lambda'(s) - \tau(s)\mu_1(s) &= 0,\\ \mu_1'(s) + \lambda(s)\tau(s) - \mu_2(s)\sigma(s) &= 0,\\ \mu_2'(s) + \mu_1(s)\sigma(s) &= 0. \end{aligned} \tag{3.19} $$

If we take the normal curve $\alpha(s)$ with constant curvatures τ and σ, the Equations (3.19) will take the form

$$ \begin{aligned} \lambda'(s) - \tau\mu_1(s) &= 0,\\ \mu_1'(s) + \tau\lambda(s) - \sigma\mu_2(s) &= 0,\\ \mu_2'(s) + \sigma\mu_1(s) &= 0. \end{aligned} \tag{3.20} $$

By differentiating the second equation of the Equation (3.20) and substituting the first and the third Equation of (3.20), we obtain the following differential equation

$$\mu_1''(s) + (\tau^2 + \sigma^2)\mu_1(s) = 0. \quad (3.21)$$

By solving the ordinary differential Equation (3.21), we obtain

$$\mu_1(s) = c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s, \quad (3.22)$$

$$\lambda(s) = \tau \int \mu_1(s)ds = \frac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_3, \quad (3.23)$$

$$\mu_2(s) = -\sigma \int \mu_1(s)ds = \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_4, \quad (3.24)$$

where c_1, c_2, c_3 and c_4 are constants.

In the following corollary, we give some characterizations for the curve to be a normal curve.

Corollary 3. *Let $\alpha(s)$ be a normal curve in Galilean 4- space G^4 with non-zero constant curvatures τ and σ. The following statements are satisfied.*

1. $<\alpha(s), N(s)> = \frac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_3$,
2. $<\alpha(s), B_1(s)> = c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s$,
3. $<\alpha(s), B_2(s)> = \frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_4$,

where c_1, c_2, c_3, and c_4 are constants.

Proof: Suppose that $\alpha(s)$ is a normal curve in Galilean 4-space G^4 with non-zero constant curvatures τ and σ, then $\alpha(s)$ can be written in the form

$\alpha(s) = (\frac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_3) N(s) + (c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s)B_1(s) + (\frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_4)B_2(s)$.

Taking the inner product of the two sides with $N(s)$, $B_1(s)$, and $B_2(s)$, the statements are held.

In the following theorem, we prove that, if $\alpha(s)$ is a normal curve, there are no curves which are congruent to $\alpha(s)$.

Theorem 2. *Let $\alpha(s)$ be a normal curve in Galilean space G^4 with non-zero constant curvatures τ and σ. Then, there are no curves which are congruent to $\alpha(s)$.*

Proof: First, let us define $m(s)$ as follows

$$m(s) = \alpha(s) - \lambda(s)N(s) - \mu_1(s)B_1(s) - \mu_2(s)B_2(s). \quad (3.25)$$

Taking the derivative of Equation (3.25) for both sides, we obtain

$m'(s) = T(s) - [\lambda'(s)N(s) + \lambda(s)N'(s) + \mu_1'(s)B_1(s) + \mu_1(s)B_1'(s) + \mu_2'(s)B_2(s) + \mu_2(s)B_2'(s)]$.

By substituting from Equations (2.1), (3.22)–(3.24).

$m'(s) = T(s) - [(c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s)N(s) + (\frac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_3)(\tau B_1(s)) + (-c_1\sqrt{\tau^2+\sigma^2} \sin\sqrt{\tau^2+\sigma^2}s + c_2\sqrt{\tau^2+\sigma^2} \cos\sqrt{\tau^2+\sigma^2}s)B_1(s)$
$+(c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s)(-\tau N(s) + \sigma B_2(s))$
$+(-c_1\sigma \cos\sqrt{\tau^2+\sigma^2}s - c_2\sigma \sin\sqrt{\tau^2+\sigma^2}s)B_2(s)$
$+((\frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s + c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_4)(-\sigma B_1(s))]$

$m'(s) = T(s) + (-\tau c_3 + \sigma c_4)B_1(s)$

$m'(s)$ does not equal to zero, which means that $m(s)$ is not a constant vector. So, $\alpha(s)$ is not congruent to a normal curve.

In the following theorem, we give the necessary condition for the normal curve in Galilean 4-space to lie on a sphere.

Theorem 3. *Let $\alpha(s)$ be a normal curve in Galilean 4-space G^4 with non-zero constant curvatures τ and σ. Then, $\alpha(s)$ lies on a sphere if $\frac{\tau}{\sigma} = \frac{c_4}{c_3}$, where c_3 and c_4 are the constants in equations (3.23) and (3.24).*

Proof: The inner product of the position vector of $\alpha(s)$ is defined by

$g(\alpha(s), \alpha(s)) = <\alpha(s), \alpha(s)>_{G^4}$
$= (\frac{\tau}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_3)^2 + (c_1 \cos\sqrt{\tau^2+\sigma^2}s + c_2 \sin\sqrt{\tau^2+\sigma^2}s)^2 + (\frac{-\sigma}{\sqrt{\tau^2+\sigma^2}}(c_1 \sin\sqrt{\tau^2+\sigma^2}s - c_2 \cos\sqrt{\tau^2+\sigma^2}s) + c_4)^2$.

By simple computations, we have

$<\alpha(s), \alpha(s)>_{G^4} = (c_1^2 + c_2^2 + c_3^2 + c_4^2) + (\frac{2c_1c_3\tau - 2c_1c_4\sigma}{\sqrt{\tau^2+\sigma^2}}) \sin\sqrt{\tau^2+\sigma^2}s + (\frac{-2c_2c_3\tau + 2c_2c_4\sigma}{\sqrt{\tau^2+\sigma^2}}) \cos\sqrt{\tau^2+\sigma^2}s$.

If $\frac{\tau}{\sigma} = \frac{c_4}{c_3}$, then $<\alpha(s), \alpha(s)>_{G^4} = (c_1^2 + c_2^2 + c_3^2 + c_4^2)$, which means that $\alpha(s)$ lies on a sphere.

4. CONCLUSION

In this study, we established the definition of the normal curve in Galilean 4- space G^4. Also, we derived the necessary condition for a curve to be a normal curve in G^4. We have proved that, if $\alpha(s)$ is a normal curve in G^4 with constant curvatures, there is no curve which is congruent to $\alpha(s)$. In the end, the necessary condition for a normal curve to lie on a sphere has been obtained.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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