Manifest Gravitational Duality Near Anti de Sitter Space-Time

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We derive a manifestly duality-invariant formulation of the Arnowitt-Deser-Misner action principle linearized around anti de Sitter background. The analysis is based on the introduction of two symmetric potentials—on which the duality transformations act—upon resolution of the linearized constraints, along the lines of previous works focusing on Minkowski and de Sitter backgrounds. Gauge freedom is crucially exploited to solve the constraints in this manner so convenient for exhibiting duality invariance, which suggests a delicate interplay between duality and gauge symmetry.

Keywords: duality, gravity, higher spin, Hamiltonian formulation of general relativity, gauge symmetry and gauge fields

1. INTRODUCTION

The understanding of dualities remains as one of the major challenges of modern theoretical physics. Dualities appear in an ample diversity of scenarios—from condensed matter physics to high energy theory—typically relating strong coupling to perturbative regimes—a rather unique feature that has played a prominent role in the elucidation of non-perturbative aspects of quantum field theory and string theory. In gravitational theories, duality has long been recognized as a constituent of the hidden symmetries that emerge upon toroidal compactifications of eleven-dimensional supergravity [1] and Einstein gravity [2, 3]. The rich algebraic structure underlying this phenomenon suggests the existence of an infinite-dimensional Kac-Moody algebra acting as a fundamental symmetry of the uncompactified theory [4–8] and encompassing the duality symmetries that appear after dimensional reduction. A characteristic property of these algebras is that they involve all the bosonic fields and their Hodge duals, including the graviton and its dual field, and so the associated symmetry transformation for a given tensor field in the bosonic sector relates it to all the rest of the fields (regardless their tensor structure) in a non-trivial way. In four dimensions, the graviton and its dual field are respectively described by symmetric tensors, and it is expected that a duality symmetry—inherited from the underlying infinite-dimensional structure—relating them may emerge. Naturally, the construction of duality-symmetric action principles constitutes an important part of the program aimed at the investigation of hidden symmetries and dualities in gravity.

In this article we show the existence of an off-shell duality symmetry in linearized gravity defined on an anti de Sitter (AdS) background, generalizing previous works where the linearization was performed on Minkowski [9] and de Sitter (dS) [10] space-times (see also [11] for the case of Maxwell theory). The analysis requires the linearization of the Arnowitt-Deser-Misner (ADM) action principle [12, 13], the choice of Poincaré coordinates for the AdS background, and the subsequent resolution of the constraints in terms of two symmetric potentials, on which the duality rotations act.

The presence of a duality symmetry in the linearized regime near an AdS background was argued in [10] on the basis of the existence of complex transformations mapping AdS into dS. Concretely, the conformally flat form of de Sitter and anti de Sitter metrics

$$d^2_{AdS} = \frac{l^2_{AdS}}{r^2}(-dt^2 + dr^2 + dx^2 + dy^2),$$

$$d^2_{dS} = \frac{l^2_{dS}}{\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2),$$

are related by the transformation $l^2_{AdS} \rightarrow il^2_{dS}$, $r \rightarrow i\eta$, $t \rightarrow iz$, the time-like boundary of AdS being mapped into a space-like boundary in dS. However, inferring the existence of a duality symmetry in the AdS case from the dS analysis [10] by this argument implies the isolation of the radial coordinate in the $3 + 1$ space-time splitting. By contrast, our analysis involves the ADM formalism and the isolation of the time-like coordinate.

We should also mention that, although our result has not a direct holographic interpretation (for we are dealing with a space-like foliation), the problem of defining duality transformations in gravity linearized around anti de Sitter background has also been addressed from the perspective of holography, motivated by the observation that there is a natural $SL(2, Z)$ action on three-dimensional conformal field theories (CFTs) with $U(1)$ conserved currents, relating the two-point function of the spin-1 conserved current of a given CFT to the two-point function of the spin-1 conserved current of a dual CFT [14]. The phenomenon was interpreted as the holographic image of the $SL(2, Z)$ electric-magnetic duality of a $U(1)$ gauge theory defined on the AdS_4 bulk. It was subsequently shown that the $SL(2, Z)$ action can be extended to two-point functions of the energy-momentum and higher spin conserved currents in three-dimensional CFTs [15, 16], a result that led the authors to conjecture that linearized higher-spin theories (including spin $s = 2$) on AdS_4 possess a generalization of electric-magnetic duality acting holographically on two-point functions on the boundary. In fact, discrete duality transformations for linearized gravity around AdS with a Pontryagin term—which acts as the analog of a theta term in electromagnetism—have been proposed in [15] using a time-like slicing of the background geometry. Despite the different character of the space-time splitting employed, it seems appropriate to keep these works in mind when seeking possible extensions of our result that include topological terms.

The rest of the article is organized as follows. In section II we derive the linearization of the ADM action principle around an anti de Sitter background, as well as the form of the gauge transformations of the canonical variables. Section III is dedicated to the resolution of the constraints in terms of potentials. In section IV we use the expression of the canonical variables in terms of potentials to construct a manifestly duality-invariant action principle. Section V summarizes our results and addresses possible extensions thereof.

2. THE LINEARIZED ADM ACTION PRINCIPLE

In order to make manifest the duality symmetry, we shall use the conformal form of the AdS metric (Poincaré coordinates):

$$ds^2 = e^\omega(dr^2 + \eta_{\alpha\beta}dx^\alpha dx^\beta), \tag{II.1}$$

where $\eta_{\alpha\beta}$ is the three-dimensional Minkowski metric, $\omega = log(l^2/r^2)$ and $l^2 = -3/\Lambda$ is the AdS radius.

Consider the ADM action principle in the presence of a cosmological constant

$$S_{ADM} = \int dt d^3x[\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N_i\mathcal{H}^i]. \tag{II.2}$$

The Hamiltonian and momentum constraints are

$$-\mathcal{H} = g^{1/2}(^{(3)}R - 2\Lambda) + g^{-1/2}(\frac{1}{2}\pi^2 - \pi^{ik}\pi^{jl}g_{ij}g_{kl}),$$

$$-\mathcal{H}^i = 2\nabla_j\pi^{ij}, \tag{II.3}$$

and the corresponding Lagrange multipliers are the lapse and shift functions

$$N = (-g^{00})^{-1/2}, \quad N_i = g_{0i}. \tag{II.4}$$

We may perform a power expansion around an AdS background as follows:

$$g_{ij} = \bar{g}_{ij} + h_{ij} + O(h^2),$$
$$\pi^{ij} = \bar{\pi}^{ij} + p^{ij} + O(p^2),$$
$$N_i = \bar{N}_i + h_{0i} + O(h^2),$$
$$N = \bar{N} - \frac{1}{2}e^{-\omega/2}h^{00} + O(h^2). \tag{II.5}$$

The bared quantities correspond to the background space-time, so $\bar{N}_i = 0$ and $\bar{N} = e^{\omega/2}$. The conjugate momentum associated to the background metric is given by

$$\bar{\pi}^{ij} = \bar{N}\bar{g}^{1/2}\left[\bar{\Gamma}^0_{kl} - \bar{g}_{kl}\bar{g}^{mn}\bar{\Gamma}^0_{mn}\right]\bar{g}^{ik}\bar{g}^{jl} = -\partial_0\omega\delta^{ij}, \tag{II.6}$$

and it vanishes in the case of an AdS background.

The linearized action principle reads

$$S[h_{ij}, p^{ij}, n, n_i] = p^{ij}\dot{h}_{ij} - H - nC - n_iC^i \tag{II.7}$$

with the Hamiltonian density

$$-H = \frac{1}{2}e^\omega p^2 - e^\omega p_{ij}p^{ij}$$

$$+\frac{3}{4}e^{-\omega}\Delta\omega h_{ij}h^{ij} + \frac{1}{2}e^{-\omega}h(\partial_i\partial_j h^{ij} - \Delta h)$$

$$+\frac{1}{2}e^{-\omega}hh^{ij}\partial_i\partial_j\omega + e^{-\omega}h\partial_i\omega\partial^i h + e^{-\omega}h^{ij}\partial_i\partial_j h$$

$$-\frac{1}{4}e^{-\omega}\partial_i h_{jk}\partial^i h^{jk} - 2e^{-\omega}\partial_i\omega h^{ij}\partial_j h - \frac{1}{4}e^{-\omega}\partial_i h\partial^i h$$

$$+e^{-\omega}\partial_i h^{ij}\partial_j h - e^{-\omega}\partial_j\omega\partial_i h^{ij}h \qquad (\text{II.8})$$

and the constraints

$$C = e^{-\omega}(\partial_i\partial_j h^{ij} - \Delta h + \partial_i\omega\partial^i h + h\Delta\omega), \qquad (\text{II.9})$$

$$C^i = \partial_j p^{ij} + \partial_j\omega p^{ij} - \frac{1}{2}\partial^i\omega p. \qquad (\text{II.10})$$

These are first-class and generate the gauge transformations

$$\delta h_{ij} = \partial_i\xi_j + \partial_j\xi_i - \xi_i\partial_j\omega - \xi_j\partial_i\omega + \delta_{ij}\partial_m\omega\xi^m$$
$$= e^{\omega}[\partial_i(e^{-\omega}\xi_j) + \partial_j(e^{-\omega}\xi_i)] + \delta_{ij}\partial_m\omega\xi^m,$$
$$\delta p_{ij} = \delta_{ij}\Delta(e^{-\omega}\xi) - \partial_i\partial_j(e^{-\omega}\xi) + \delta_{ij}\partial_l\omega\partial^l(e^{-\omega}\xi). \qquad (\text{II.11})$$

The Lagrange multipliers have been defined as $n_i = -2h_{0i}$ and $n = -\frac{1}{2}h^{00}$. The equations of motion for the background metric (A.14 in **Appendix**) have been used. Indices are raised and lowered with the flat spatial metric η_{ij}.

3. RESOLUTION OF THE CONSTRAINTS

We notice that, in order to solve the constraints (II.10) in terms of potentials, it is convenient to perform specific gauge transformations that render them in a form similar to the flat background case. Consider the gauge choice

$$h_{ij} = j_{ij} + e^{\omega}[\partial_i(e^{-\omega}v_j) + \partial_j(e^{-\omega}v_i)] + \delta_{ij}\partial_m\omega v^m,$$
$$p^{ij} = q^{ij} + \delta^{ij}\Delta u - \partial^i\partial^j u + \delta^{ij}\partial_k\omega\partial^k u, \qquad (\text{III.1})$$

where j_{ij} satisfies $\partial_i\omega\partial^i j + \Delta\omega j = \partial^i(\partial_i\omega j) = 0$ and q^{ij} is traceless. To prove the existence of such a gauge, it is sufficient to find two particular functions v_i and u verifying

$$\partial^i(\partial_i\omega h) = \partial^i[\partial_i\omega(2\partial_m v^m + \partial_m\omega v^m)] \qquad (\text{III.2})$$

and

$$p = 2\Delta u + 3\partial_m\omega\partial^m u. \qquad (\text{III.3})$$

The following choice fulfills the previous requirements:

$$v_i = \partial_i\Delta^{-1}(e^{\omega/2}\frac{h - f(t,x,y)(\partial_r\omega)^{-1}}{2}),$$
$$u = \frac{e^{-3\omega/4}}{2}[\Delta - \frac{15}{8}\Delta\omega]^{-1}[e^{3\omega/4}p]. \qquad (\text{III.4})$$

where $f(t,x,y)$ is a function independent of the radial coordinate r, obtained from the integration of (III.2). In the sequel we shall not specify a particular form for the functions u and v_i: they will be treated as scalar and vector potentials, respectively.

The constraints now read

$$e^{-\omega}(\partial_i\partial_j j^{ij} - \Delta j) = 0, \qquad (\text{III.5})$$
$$e^{-\omega}\partial_j(e^{\omega}q^{ij}) = 0, \qquad (\text{III.6})$$

and remain invariant under the residual gauge transformations

$$\delta j_{ij} = \partial_i\chi_j + \partial_j\chi_i, \qquad (\text{III.7})$$
$$\delta q^{ij} = e^{-\omega}(\delta^{ij}\Delta\chi - \partial^i\partial^j\chi). \qquad (\text{III.8})$$

We may use the residual gauge freedom (III.7) to carry away the trace of j_{ij}. This is clearly consistent with the previous gauge choice (III.2). The constraint (III.5) is then solved in terms of potentials as follows:

$$j_{ij} = \epsilon_{iab}\partial^a\phi_j^b + \epsilon_{jab}\partial^a\phi_i^b + \partial_i w_j + \partial_j w_i, \qquad (\text{III.9})$$

for some vector potential w_i. On the other hand, the residual gauge freedom (III.8) may be used to write q^{ij} –constrained to obey $q = 0$– in terms of an unconstrained variable k^{ij} defined as

$$q^{ij} = k^{ij} + e^{-\omega}(\delta^{ij}\Delta s - \partial^i\partial^j s) \qquad (\text{III.10})$$

for some function s such that $k = -2e^{-\omega}\Delta s$. The constraint (III.6) is solved as follows:

$$q^{ij} = e^{-\omega}\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl} + e^{-\omega}(\delta^{ij}\Delta s - \partial^i\partial^j s). \qquad (\text{III.11})$$

An alternative way to derive the previous expression is to first solve (III.6) in terms of a constrained potential Q_{ij}

$$q^{ij} = e^{-\omega}\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k Q_{nl} \qquad (\text{III.12})$$

and then write $Q^{ij} = P^{ij} + T^{ij}$ for some unconstrained potential P^{ij} and some tensor $T^{ij} = T^{ij}(P)$ constructed to obey $\epsilon^{imn}\epsilon_i^{\ kl}\partial_m\partial_k P_{nl} = \epsilon^{imn}\epsilon_i^{\ kl}\partial_m\partial_k T_{nl}$ and to generate a gauge transformation of the form (III.8). The particular choice $T^{ij} = \frac{1}{2}\delta^{ij}(P - \partial_a\partial_b\Delta^{-1}P^{ab})$ fulfills these conditions.

The final expressions for the canonical variables are

$$h_{ij} = \epsilon_{iab}\partial^a\phi_j^b + \epsilon_{jab}\partial^a\phi_i^b + \partial_i w_j + \partial_j w_i$$
$$+ e^{\omega}[\partial_i(e^{-\omega}v_j) + \partial_j(e^{-\omega}v_i)] + \delta_{ij}\partial_m\omega v^m, \qquad (\text{III.13})$$

$$p^{ij} = e^{-\omega}\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl} + e^{-\omega}(\delta^{ij}\Delta s - \partial^i\partial^j s)$$
$$+ \delta^{ij}\Delta u - \partial^i\partial^j u + \delta^{ij}\partial_k\omega\partial^k u. \qquad (\text{III.14})$$

As observed in the case of Minkowski and de Sitter backgrounds, there is an ambiguity in the definition of the potentials determined by the equations

$$\delta h_{ij} = \epsilon_{iab}\partial^a\delta\phi_j^b + \epsilon_{jab}\partial^a\delta\phi_i^b + \partial_i\delta w_j + \partial_j\delta w_i$$
$$+ e^\omega[\partial_i(e^{-\omega}\delta v_j) + \partial_j(e^{-\omega}\delta v_i)] + \delta_{ij}\partial_m\omega\delta v^m$$
$$= \partial_i\xi_j + \partial_j\xi_i - \xi_i\partial_j\omega - \xi_j\partial_i\omega + \delta_{ij}\partial_m\omega\xi^m$$
$$\text{(III.15)}$$

and

$$\delta p^{ij} = e^{-\omega}\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k\delta P_{nl} + e^{-\omega}(\delta^{ij}\Delta\delta s - \partial^i\partial^j\delta s)$$
$$+ \delta^{ij}\Delta\delta u - \partial^i\partial^j\delta u + \delta^{ij}\partial_k\omega\partial_k\delta u$$
$$= \delta^{ij}\Delta(e^{-\omega}\xi) - \partial^i\partial^j(e^{-\omega}\xi) + \delta^{ij}\partial_l\omega\partial^l(e^{-\omega}\xi).$$
$$\text{(III.16)}$$

They are solved as follows:

$$\delta\phi_{ij} = \partial_i\alpha_j + \partial_j\alpha_i + \delta_{ij}\beta,$$
$$\delta P_{ij} = \partial_i\gamma_j + \partial_j\gamma_i + \delta_{ij}\eta,$$
$$\delta v_i = \xi_i,$$
$$\delta u = e^{-\omega}\xi,$$
$$\delta w_i = -\epsilon_{iab}\partial^a\alpha^b,$$
$$\delta s = -\eta.$$
$$\text{(III.17)}$$

4. MANIFEST DUALITY INVARIANCE

In this section, we shall use the expression of the canonical variables in terms of the potentials to cast the action principle in a manifestly duality-invariant form. Let us focus first on the kinetic term. Written in terms of the potentials, it reads

$$p^{ij}\dot{h}_{ij} = e^{-\omega}\epsilon^{imn}\epsilon^{jkl}\epsilon_{iab}\partial_k\partial_m P_{nl}\partial^a\dot{\phi}_j^b. \qquad \text{(IV.1)}$$

The action of the duality transformation $P_{ij} \rightarrow \phi_{ij}$, $\phi_{ij} \rightarrow -P_{ij}$ on the kinetic term yields (up to total derivatives)

$$S_K \rightarrow S_K - \int dt d^3x \partial_k\omega\epsilon^{imn}\epsilon^{jkl}\epsilon_i^{ab}e^{-\omega}\partial_m\dot{P}_{nl}\partial_a\phi_{bj}. \qquad \text{(IV.2)}$$

The crucial observation is that the extra term in (IV.2) can be written as a sum of total derivatives:

$$-\partial_k\omega\epsilon^{imn}\epsilon^{jkl}\epsilon_i^{ab}e^{-\omega}\partial_m\dot{P}_{nl}\partial_a\phi_{bj} =$$
$$\epsilon^{imn}\epsilon^{jkl}\epsilon_i^{ab}\{-\partial_m[\partial_k\omega e^{-\omega}\dot{P}_{nl}\partial_a\phi_{bj}]$$
$$-\partial_m[e^{-\omega}\partial_a\omega\dot{P}_{nl}\partial_k\phi_{bj}] + \partial_k\partial_a[e^{-\omega}\dot{P}_{nl}\partial_m\phi_{bj}]$$
$$+\frac{1}{2}\partial_m[\partial_k\omega\partial_a\omega e^{-\omega}\dot{P}_{nl}\phi_{bj}] + \partial_a[\partial_k\partial_m(e^{-\omega}\dot{P}_{nl})\phi_{bj}]$$
$$-\partial_a[\partial_k\partial_m(e^{-\omega}\phi_{bj})\dot{P}_{nl}] + \partial_k\partial_m[\partial_a\omega e^{-\omega}\dot{P}_{nl}\phi_{bj}]$$
$$+\partial_a\partial_m[\partial_k\omega e^{-\omega}\dot{P}_{nl}\phi_{bj}] - \partial_k\partial_a[\partial_m\omega e^{-\omega}\dot{P}_{nl}\phi_{bj}]$$

$$-\partial_k\partial_m[e^{-\omega}\partial_a\dot{P}_{nl}\phi_{bj}] - \partial_a\partial_m[e^{-\omega}\partial_k\dot{P}_{nl}\phi_{bj}]$$
$$+ \partial_k\partial_a[e^{-\omega}\partial_m\dot{P}_{nl}\phi_{bj}]\}. \qquad \text{(IV.3)}$$

Therefore, the kinetic term is invariant under duality transformation (up to total derivatives). The argument can be extended to show the invariance of S_K under $SO(2)$ duality rotations (again, up to total derivatives).

On the other hand, substitution of (III.14) in the Hamiltonian density (II.8) yields:

$$-H = e^{-\omega}[-\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl}\epsilon_{ipq}\epsilon_{jrs}\partial^p\partial^r P^{qs}$$
$$-\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k\phi_{nl}\epsilon_{ipq}\epsilon_{jrs}\partial^p\partial^r\phi^{qs}$$
$$+\frac{1}{2}(\epsilon^{imn}\epsilon_i^{kl}\partial_m\partial_k P_{nl})^2 + \frac{1}{2}(\epsilon^{imn}\epsilon_i^{kl}\partial_m\partial_k\phi_{nl})^2]$$
$$-e^{-\omega}[\partial_i\partial_j\phi_{kl}\partial^i\partial^j\phi^{kl} - \partial_i\partial_j\phi_{kl}\partial^k\partial^j\phi^{il}$$
$$+\partial^i\partial_j\phi_{ik}\partial^j\partial^k\phi - \frac{1}{2}\partial_i\partial_j\phi\partial^i\partial^j\phi]$$
$$+e^{-\omega}[\partial^i\partial_j\phi_{ik}\partial^j\partial_l\phi^{kl} - \frac{1}{2}\partial^i\partial^j\phi_{ki}\partial^k\partial^l\phi_{jl}]$$
$$+3\Delta\omega[\partial_i\phi_{jk}\partial^i\phi^{jk} - \partial_i\phi_{jk}\partial^j\phi^{ik} - \frac{1}{2}\partial^j\phi_{jk}\partial_i\phi^{ik}$$
$$+\partial^i\phi_{ij}\partial^j\phi - \frac{1}{2}\partial^i\phi\partial_i\phi] + \frac{1}{2}e^{-\omega}\partial^i\omega[\partial^j\phi_{ik}\partial_j\partial_l\phi^{kl}$$
$$-\partial_j\phi_{ik}\partial^k\partial_l\phi^{jl} + 2\partial^j\phi_{jk}\partial^k\partial^l\phi_{il} - 2\partial_j\phi\partial^j\partial^k\phi_{ik}$$
$$-\partial^j\phi_{jk}\partial_i\partial_l\phi^{lk} + \partial_j\phi\partial_i\partial_k\phi^{jk} + \partial_j\phi^{lk}\partial^j\partial_i\phi_{lk}$$
$$-\partial^j\phi^{kl}\partial_i\partial_l\phi_{jk} - \partial_i\phi^{jk}\Delta\phi_{jk} + \partial_j\phi_{ik}\Delta\phi^{jk}$$
$$+3\partial^j\phi^{kl}\partial_j\partial_k\phi_{il} - 3\partial^j\phi^{kl}\partial_k\partial_l\phi_{ji} - \partial_j\phi^{jk}\Delta\phi_{ik}$$
$$+\partial^j\phi\Delta\phi_{ij} + \partial_i\phi^{jk}\partial_j\partial_k\phi - \partial_j\phi_{ik}\partial^j\partial^k\phi]. \qquad \text{(IV.4)}$$

After integration by parts, the Hamiltonian density can be cast in a more symmetric form:

$$-H = e^{-\omega}[-\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k P_{nl}\epsilon_{ipq}\epsilon_{jrs}\partial^p\partial^r P^{qs}$$
$$-\epsilon^{imn}\epsilon^{jkl}\partial_m\partial_k\phi_{nl}\epsilon_{ipq}\epsilon_{jrs}\partial^p\partial^r\phi^{qs}$$
$$+\frac{1}{2}(\epsilon^{imn}\epsilon_i^{kl}\partial_m\partial_k P_{nl})^2$$
$$+\frac{1}{2}(\epsilon^{imn}\epsilon_i^{kl}\partial_m\partial_k\phi_{nl})^2] + e^{-\omega}V \qquad \text{(IV.5)}$$

with

$$V = 3\Delta\omega[\partial_i\phi_{jk}\partial^i\phi^{jk} - \partial_i\phi_{jk}\partial^j\phi^{ik} - \frac{1}{2}\partial^j\phi_{jk}\partial_i\phi^{ik}$$
$$+\partial_i\phi^{ij}\partial_j\phi - \frac{1}{2}\partial^i\phi\partial_i\phi] + \frac{1}{2}\partial^i\omega[-\partial^k\phi_{ji}\partial_k\partial_l\phi^{jl}$$
$$-\partial^j\phi_{ik}\partial^k\partial^l\phi_{jl} + 2\partial^j\phi_{jk}\partial^k\partial^l\phi_{li} - 2\partial^k\phi\partial_k\partial^l\phi_{il}$$
$$+7\partial_j\phi^{jk}\partial_i\partial^l\phi_{kl} - 3\partial_k\phi\partial_i\partial_j\phi^{kj} + \partial^k\phi_{ji}\partial_i\partial_k\phi^{jl}$$
$$-\partial_j\phi_{lk}\partial_i\partial^l\phi^{jk} + \partial_i\phi^{jk}\Delta\phi_{jk} + \partial_j\phi_{ik}\Delta\phi^{jk}$$
$$+5\partial_l\phi^{jk}\partial_k\partial^l\phi_{ij} - 3\partial^j\phi^{lk}\partial_l\partial_k\phi_{ji} - 9\partial_k\phi^{jk}\Delta\phi_{ji}$$
$$+5\partial^j\phi\Delta\phi_{ji} + \partial_i\phi^{jk}\partial_j\partial_k\phi - \partial_j\phi_{ki}\partial^j\partial^k\phi$$

$$+3\partial^j\phi\partial_i\partial_j\phi - 3\partial_i\phi\Delta\phi + 3\partial_k\phi^{jk}\partial_j\partial^l\phi_{il}$$
$$-3\partial^k\phi_{ik}\partial_j\partial_l\phi^{jl} - 6\partial_k\phi^{jk}\partial_i\partial_j\phi + 6\partial^j\phi_{ij}\Delta\phi].$$

$$\text{(IV.6)}$$

One can show that the term $e^{-\omega}V$ is a sum of total derivatives, similarly to what we have found in (IV.3). The $SO(2)$ duality invariance of the action principle is now manifest.

5. CONCLUSIONS

We have shown that linearized gravity around anti de Sitter space-time can be cast in a manifestly duality-invariant form upon resolution of the ADM constraints in terms of two symmetric potentials. The analysis relies on the use of Poincaré coordinates for the AdS background metric. Gauge freedom is exploited in order to introduce the two symmetric potentials in the resolution of the constraints, which suggests a close relationship and interplay between duality and gauge symmetry. This result complements previous works where the linearization was performed around Minkowski and de Sitter space-times, and allows us to conclude that $SO(2)$ duality is a symmetry of the linearized ADM action around maximally symmetric backgrounds. The structure of the duality-symmetric action principle is similar in the three cases after integrating by parts and dropping boundary terms, the only difference being background-dependent relative factors in the kinetic term and the Hamiltonian. The potentials enjoy the same gauge invariances in the three cases.

We have found that duality transformations leave invariant the action principle up to the addition of surface terms on the space-like boundary of AdS. An analogous phenomenon lies at the root of the duality conjecture [15, 16] in holography: the introduction of surface terms in the time-like boundary typically requires the modification of boundary conditions and, since modified boundary conditions are associated with deformations of boundary CFTs, the action of duality in the bulk would imply a transformation of the CFT.

An important feature of the potential formalism, which we have also encountered in the present article, is the absence of manifest space-time covariance. Although in some instances it is possible to recover manifest space-time covariance for duality-symmetric action principles (either by the introduction of an infinite number of auxiliary fields [17–20] with polynomial dependence or a finite number of auxiliary fields with non-polynomial dependence [21, 22]), when it comes to the case of gravity one may argue that this will probably not be the case by plain contrast of two well-known results. On the one hand (a discrete version of), electric-magnetic duality is consistent with quantum mechanics [23, 24]. On the other hand, the notion of manifest space-time covariance seems to be inconsistent with the quantum dynamics of gravity [25, 26]. The immediate conclusion is that, at least in a background-independent approach to quantum gravity, a discrete version of electric-magnetic duality

would be allowed, while manifest space-time covariance would not.

Last, let us mention possible extensions of the present work. Along the lines of [16], it would be interesting to consider the inclusion of topological terms in the action principle, in particular the Pontryagin term, then determine whether the constraints are still solvable in terms of potentials and finally search for a [perhaps $SL(2, Z)$] duality-invariant formulation of the action principle. The potential analysis could likewise be performed in the case of a time-like foliation, as a complement to [16]. The derivation of the twisted self-duality equations of motion also deserves investigation, including possible connections with the parent action method for the construction of dual Lagrangians [27, 28]. The generalization of our work to the case of arbitrary higher spin fields coupled to a fixed AdS background should as well be studied, building on the works [29] and [30, 31]. Finally, it would be interesting to study how the inclusion of boundary counterterms [32–34] in AdS affects the potential analysis.

DEDICATION

This article is dedicated to the memory of Rosario "Charo" Aranda.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/**Supplementary Material**.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2019.00188/full#supplementary-material

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