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Commentary: A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel-Wright Function

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A Commentary on

A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel-Wright Function
by Agarwal, R., Jain, S., Agarwal, R. P., and Baleanu, D. (2018). Front. Phys. 6:79. doi: 10.3389/fphy.2018.00079

1. PRELIMINARIES

The commented paper [1] is one example of a long list of recently published works devoted to evaluation of the images of classes of special functions under the classical operators of fractional order integration and differentiation and their various generalizations. The used procedures are usually same standard ones [interchanging the order of integration and summation, as in (2.3), [1]]. And in many cases the resulting expressions are complicated and involve special functions different in kind from the originals. Since both *Special Functions* (*SF*) and operators of *Fractional Calculus* (*FC*) are of great variety, the mentioned publications are growing daily, and their authors seem not observing the fact that *the problem has been resolved in its generality rather earlier*. The main results behind a general approach have been proposed by Kiryakova since 1994 and on, see for example: [2, p.20–1, n; Th.5.2.1], [3, L.3.2, L.4.4], [4, Equations (9), (16)], [5, Th.9], etc. And for the inspiring ideas about, we need to pay tribute to some classical authors in FC and SF, to mention just a few their works as Erdélyi [6], Lavoie et al. [7], Marichev [8], and Prudnikov et al. [9], etc. To remind the audience the known facts, and to propose an alternative approach saving efforts in particular cases, recently the author of this commentary published the surveys [10] and [11]. The references therein contain only a very small number of the recent publications on the subject.

To start with, a classical result (say from Erdélyi [6, §13.1, (95)], Lavoie et al. [7, Table 17.1, p. 261]) says that a Riemann-Liouville fractional integral maps (in general) a ${}_pF_q$-function into a ${}_{p+1}F_{q+1}$-function, increasing its indices by one. The same result is valid for the more general *Erdélyi-Kober* (*EK*) *operator of fractional integration*. For convenience, in the (left-hand sided) operators of FC as EK, instead of $\int_0^x$ we use a representation by $\int_0^1$ after a substitution $t/x = \sigma$ (see [2]):

$$I_\beta^{\gamma,\delta} f(x) = \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1}\, t^{\beta\gamma} f(t) d(t^\beta)$$
$$= \frac{1}{\Gamma(\delta)} \int_0^1 \sigma^\gamma\, (1-\sigma)^{\delta-1} f(x\sigma^{\frac{1}{\beta}})\, d\sigma,$$
$$\text{with real } \delta > 0, \gamma, \beta > 0. \tag{1}$$

Note that the Riemann-Liouville (RL) integral comes from (1) for $\gamma = 0, \beta = 1$, and the EK operator (1.26) in Agarwal et al. [1] is for $\beta = 1$. For example, for $p \leq q$ or $p = q$ (then $|x| < 1$), the result in Kiryakova [2, (4.2.2$'$), p.218], [3, L.3.2], reads: $I_1^{a_{p+1}-1, b_{q+1}-a_{p+1}} \left\{ {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; \lambda x) \right\} = [\Gamma(b_{q+1})/\Gamma(a_{p+1})]\, {}_{p+1}F_{q+1}(a_1, ..., a_p, a_{p+1}; b_1, ..., b_q, b_{q+1}; \lambda x)$. For the *generalized Wright hypergeometric function* this relation was easily extended (see e.g., Kiryakova [5, L.7] and [10, L.1]) to

$$I_\beta^{\gamma,\delta} \left\{ {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \middle| \lambda x^\mu \right] \right\}$$
$$= {}_{p+1}\Psi_{q+1} \left[\begin{array}{c} (a_i, A_i)_1^p, (\gamma+1, 1/\beta) \\ (b_j, B_j)_1^q, (\gamma+\delta+1, 1/\beta) \end{array} \middle| \lambda x^\mu \right]. \tag{2}$$

Next, we need to give a short background on the operators of *Generalized Fractional Calculus* (*GFC*), Kiryakova [2]. The basic definition of the operator of *generalized fractional integration* (*generalized fractional integral, multiple EK integral*) uses a single integral representation with a special function in the kernel: a Meijer's $G^{m,0}_{m,m}$-function in the simpler case (all $\beta_1 = ... = \beta_m = \beta$) or a Fox's $H^{m,0}_{m,m}$-function (in general), where: integer $m \geq 1$ is the "multiplicity," ($\delta_1 \geq 0, ..., \delta_m \geq 0$)—the multi-order of integration, $(\gamma_1, ..., \gamma_m)$ - a real multi-weight and ($\beta_1 > 0, ..., \beta_m > 0$)—additional multi-parameter. Namely,

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) := \int_0^1 H^{m,0}_{m,m} \left[\sigma \left| \begin{array}{c} (\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \\ (\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k})_1^m \end{array} \right. \right]$$
$$f(x\sigma) d\sigma, \text{ if } \sum_{k=1}^m \delta_k > 0, \tag{3}$$

and it is the identity operator if $\delta_1 = \delta_2 = \cdots = \delta_m = 0$. In the case of all equal β's: $\beta_1 = \beta_2 = ... = \beta_m = \beta > 0$, (3) has a simpler representation by means of *Meijer's G-function*,

$$I^{(\gamma_k),(\delta_k)}_{(\beta,...,\beta),m} f(x) := I^{(\gamma_k),(\delta_k)}_{\beta,m} f(x)$$
$$= \int_0^1 G^{m,0}_{m,m} \left[\sigma \left| \begin{array}{c} (\gamma_k + \delta_k)_1^m \\ (\gamma_k)_1^m \end{array} \right. \right] f(x\sigma^{1/\beta}) d\sigma$$
$$= \left[\prod_{k=1}^m I_\beta^{\gamma_k,\delta_k} \right] f(x). \tag{4}$$

The fact which explains the wide applications of (3) and (4) and their numerous particular cases, is that the same operators can be represented also without any use of SF, by means of the repeated integral representations for the *commutative products of classical EK operators* (1), see [2, Ch.1, Ch.5]:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) := \left[\prod_{k=1}^m I_{\beta_k}^{\gamma_k,\delta_k} \right] f(x)$$
$$= \int_0^1 \cdots \int_0^1 \left[\prod_{k=1}^m \frac{(1-\sigma_k)^{\delta_k - 1} \sigma_k^{\gamma_k}}{\Gamma(\delta_k)} \right]$$
$$f\left(x \sigma_1^{1/\beta_1} \ldots \sigma_m^{1/\beta_m} \right) d\sigma_1 \ldots d\sigma_m. \tag{5}$$

A general result concerning the topic of the commented paper [1], as well as of dozens of similar papers, is stated in our survey papers [10, Th.1, (45)] and [11, Th.4.1], and reads as follows: *The image of a generalized Wright hypergeometric function under GFC integral* (3)–(5), is the *same* kind of SF but with indices increased by the multiplicity $m \geq 1$ (and additional parameters):

$$I^{(\gamma_k)_1^m,(\delta_k)_1^m}_{(\beta_k)_1^m,m} \left\{ x^c\, {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{array} \middle| \lambda x^\mu \right] \right\}$$
$$= x^c\, {}_{p+m}\Psi_{q+m} \left[\begin{array}{c} (a_i, A_i)_1^p, (\gamma_i + 1 + \frac{c}{\beta_i}, \frac{\mu}{\beta_i})_1^m \\ (b_j, B_j)_1^q, (\gamma_i + \delta_i + 1 + \frac{c}{\beta_i}, \frac{\mu}{\beta_i})_1^m \end{array} \middle| \lambda x^\mu \right]. \tag{6}$$

No need to explain that this result comes by combination of (2) and (5), by m successive steps. Similar result holds for the generalized fractional differentiation $D^{(\gamma_k)_1^m,(\delta_k)_1^m}_{(\beta_k)_1^m,m}$, [2], [10], [11]. Explanations on the procedures, and important corollaries in the 3 cases $p < q$, $p = q$, $p = q + 1$, have been given in details yet in Kiryakova [2, Ch.4], [3]—for the images of the ${}_pF_q$-functions, and in Kiryakova [5]—for ${}_p\Psi_q$. For definitions of the mentioned Special Functions, one can refer e.g., to Erdélyi [6] and Prudnikov et al. [9].

2. THE CASE IN THE COMMENTED PAPER

First, let us comment on the so-called *Marichev-Saigo-Maeda* (*MSM*) *operator* of FC (for shortness of place we consider only the left-hand sided I_{0+} version), as it is defined by (1.10) in Agarwal et al. [1], and also after a substitution $\sigma = t/x$, as:

$$I^{\xi,\xi',\rho,\rho',\varkappa} f(x)$$
$$= \frac{x^{-\xi}}{\Gamma(\varkappa)} \int_0^x (x-t)^{\varkappa-1} t^{-\xi'} F_3(\xi, \xi', \rho, \rho'; \varkappa; 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt$$
$$= x^{\varkappa-\xi-\xi'} \int_0^1 \frac{(1-\sigma)^{\varkappa-1}}{\Gamma(\varkappa)} \sigma^{-\xi'} F_3(\xi, \xi', \rho, \rho'; \varkappa;$$
$$1-\sigma, 1-\frac{1}{\sigma}) f(x\sigma) d\sigma, \quad \Re\varkappa > 0, \text{ etc. conditions.} \tag{7}$$

The kernel special function F_3, known as the *Appell function* of 2 variables (or *Horn function*) is defined by the power series [see (1.6)]

$$F_3(\xi,\xi',\rho,\rho';\varkappa;x,y) = \sum_{m,n=0}^{\infty} \frac{(\xi)_m(\xi')_n(\rho)_m(\rho')_n}{(\varkappa)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \quad \max\{|x|,|y|\} < 1,$$

but as shown in [9, § 8.4.51, (2)] and mentioned in our works as Kiryakova [2, Ch.1], [4], etc., it is *also a Meijer G-function*,

$$\frac{(1-\sigma)^{\varkappa-1}}{\Gamma(\varkappa)} \sigma^{-\xi'} F_3(\xi,\xi',\rho,\rho';\varkappa;1-\sigma,1-\frac{1}{\sigma}) = G^{3,0}_{3,3}\left[\sigma \,\middle|\, \begin{matrix} \xi'+\rho', \varkappa-\xi, \varkappa-\rho \\ \xi',\rho',\varkappa-\xi-\rho \end{matrix}\right]. \quad (8)$$

Note that both above expressions of F_3 are symmetric w.r.t. the 2 variables, to ξ and ξ', resp. ρ and ρ', and w.r.t. the parameters in upper and bottom rows in G-function (8). In view of (4), this comes to say that *the MSM fractional integral is nothing but an operator of our GFC with multiplicity* $m = 3$ and $\beta = 1$, and so, also a composition (5) of 3 commuting classical EK fractional integrals (1). In Kiryakova [2, p. 21], [10, (64)], etc. various such representations of (7) are provided, as the mentioned symmetry allows this. Here, for the purpose to compare with the results in Agarwal et al. [1], we mention the following representation in terms of (4) and (3):

$$\begin{aligned} &I^{\xi,\xi',\rho,\rho',\varkappa}_{0+} f(x) \\ &= x^{\varkappa-\xi-\xi'} \int_0^1 G^{3,0}_{3,3}\left[\sigma \,\middle|\, \begin{matrix} \rho', \varkappa-\xi-\xi', \varkappa-\xi'-\rho \\ 0, \varkappa-\xi-\xi'-\rho, -\xi'+\rho' \end{matrix}\right] f(x\sigma)d\sigma \\ &= I^{(0,\varkappa-\xi-\xi'-\rho,-\xi'+\rho'),(\rho',\rho,\varkappa-\rho-\rho')}_{(1,1,1),3} f(x), \end{aligned} \quad (9)$$

and thus also, as the composition of three (commuting) EK integrals: $I^{\xi,\xi',\rho,\rho',\varkappa}_{0+} = I^{0,\rho'}_1 \, I^{\varkappa-\xi-\xi'-\rho,\rho}_1 \, I^{-\xi'+\rho',\varkappa-\rho-\rho'}_1$.

Then, it is a turn to consider the nature of the *generalized Lommel-Wright function* treated in the commented paper [1]. There, we find its definition (1.1) and also a representation as a *generalized Wright hypergeometric function* (1.2), namely:

$$\begin{aligned} J^{\varphi,m}_{\omega,\theta}(z) &= (\frac{z}{2})^{\omega+2\theta} \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{z}{2})^{2k}}{\big(\Gamma(\theta+k+1)\big)^m \, \Gamma(\omega+k\varphi+\theta+1)} \\ &= (\frac{z}{2})^{\omega+2\theta} \, {}_1\Psi_{m+1}\big[(1,1);(\theta+1,1),...,(\theta+1,1), \\ &\quad (\omega+\theta+1,\varphi); -z^2/4\big]. \end{aligned} \quad (10)$$

Additionally, we are tempted to mention that (10) is a typical example of the *multi-index Mittag-Leffler function* (see e.g., [10], etc.), namely $J^{\varphi,m}_{\omega,\theta}(z) = (\frac{z}{2})^{\omega+2\theta} E^{(m+1)}_{(1,...,1,\varphi),(\theta+1,...,\theta+1,\omega+\theta+1)}\big(-(\frac{z}{2})^2\big)$. Note also that the MSM image of *arbitrary* ${}_p\Psi_q$-function is given in Cor. 4, Equation (65) of Kiryakova [10].

But specially in this case, it remains to combine the general result (6) in the case $m = 3$ and suitable parameters as for the MSM operator (9) with the representation of (10) as a generalized Wright hypergeometric function ${}_1\Psi_{m+1}$. As expected, its MSM image, as a composition of three EK fractional integrals, gives again a generalized Wright hypergeometric function but with indices increased by 3, that is, a generalized Wright hypergeometric function ${}_4\Psi_{m+4}$-function, *as is the correctly evaluated but predictable result* (2.2) of Theorem 2.1 in Agarwal et al. [1]:

$$\begin{aligned} &I^{\xi,\xi',\rho,\rho',\varkappa}_{0+}\left[t^{\chi-1} J^{\varphi,m}_{\omega,\theta}(tz)\right](x) = x^{A-\xi-\xi'+\varkappa-1}(\frac{z}{2})^{\omega+2\theta} \\ &\times {}_4\Psi_{4+m}\left[\begin{matrix} (A,2),(A+\varkappa-\xi-\xi'-\rho,2),(A+\rho'-\xi',2),(1,1) \\ (A+\rho',2),(A+\varkappa-\xi-\xi',2),(A+\varkappa-\xi'-\rho,2),(\omega+\theta+1,\varphi),(\theta+1,1) \end{matrix} \,\middle|\, -\frac{(zx)^2}{4}\right], \text{ with } A := \chi+\omega+2\theta. \end{aligned} \quad (11)$$

What concerns the result (3.2) of Theorem 3.1, [1] for "image formulas for *Beta transform*" (1.28), as *many other authors also* call this integral operator, note that in terms of FC it is just a very special case of the Erdélyi-Kober fractional integral (1), namely:

$$B\left\{f(xt);a,b\right\} = \int_0^1 t^{a-1}(1-t)^{b-1}f(xt)dt = \Gamma(b)\left\{I^{a-1,b}_1 f\right\}(x). \quad (12)$$

Thus, application of the Beta transform (single EK integral, $m = 1, \beta = 1$) to the MSM operator (which is 3-tuple EK, generalized fractional integral, $m = 3$), leads practically to application of a composition of 4 classical EK integrals, and again under the philosophy of our general result (6), this maps a ${}_P\Psi_Q$-function into a ${}_{P+4}\Psi_{Q+4}$-function. In the case of the generalized Lommel-Wright function (10) (i.e., ${}_1\Psi_{1+m}$), it is *predictable* (in sense of survey [10]) that the result is given by the ${}_5\Psi_{5+m}$-function in (3.2) of Agarwal et al. [1]. The results for the right-hand sided operators I_{0-} and for all mentioned special cases of operators, including Saigo operators (1.16) (cases of GFC with $m = 2$), the particular case (1.26) with $\beta = 1$ of EK operators (1), etc. and for various special functions (Bessel, Struve, Lommel-Wright, etc.) come from the general result and are presented as some illustrating examples in Kiryakova [10], [11]. Thus, claims in Abstract of Agarwal et al. [1] sound ambitious. For the *pathway P_δ-transform* (1.30) (introduced and studied initially by Mathai and Haubold), from Th.9, (43) in Kumar cited in Agarwal et al. [1] as [24], one can immediately derive the result (3.7) of Th.3.2 in Agarwal et al. [1], since a P_δ-transform of a ${}_P\Psi_Q$ [in this case ${}_4\Psi_{4+m}$ from (11)] gives a ${}_{P+1}\Psi_Q$-function (i.e., ${}_5\Psi_{4+m}$).

3. CONCLUSIONS

One can advise the researchers on the topic to follow a procedure like this:

1) Check if the treated SF can be presented as a Wright g.h.f. ${}_p\Psi_q$;
2) Check if the operator of FC to be evaluated can be presented as composition of classical EK operators (i.e., also in the form (3);
3) If so (usually it is the case), apply a general result like (6);
4) Most visible results would be to present a FC image of a SF in terms of *same kind* of SF (with only changed indices and parameters).

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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