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How Nonassociative Geometry Describes a Discrete Spacetime

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Nonassociative geometry, providing a unified description of discrete and continuum spaces, is a valuable candidate for the study of discrete models of spacetime. Within the framework of nonassociative geometry we propose a model of emergent spacetime. In our approach, the evolution of spacetime geometry is governed by a random/stochastic process. This leads to a natural appearance of causal structure and arrow of time. We apply our approach to study a toy model of discrete (2+1)-D spacetime and Friedmann-Robertson-Walker cosmological model. We show that in a continuous limit the evolution of the discrete spacetime corresponds to a radiation epoch of the standard cosmological model.

Keywords: nonassociative geometry, discrete spacetime, quasigroups, smooth loops, quantum gravity

1. INTRODUCTION

Any attempt to create a quantum gravitation theory faces the challenge of comprehension the concept of quantizing of spacetime and of describing the quantum nature of space and time [1–3]. As part of the general problem of time, one of the most challenging and open problems is the origin of irreversibility in our universe, also known as the problem of the arrow of time [1, 4–9].

One can distinguish two general strategies to achieve this goals [10, 11]. The first strategy consists in quantizing a classical structure, that later is recovered as a limit of the quantum theory. The second strategy assumes that the classical structures are emergent from the other theory, which is more fundamental theory from the very beginning. In the second approach, a formulation of the quantum theory may require omitting use of continuum concepts *a priori*. This means that at the Planck scale the standard concept of spacetime must be replaced by some discrete structure [10, 12].

The proposal that spacetime may be fundamentally discrete has been adopted by numerous strategies [13, 14]. This discreteness is motivated by several heuristic arguments, but so far has not been deduced rigorously. In Nesterov and Sabinin [15–18], we have proposed a new unified approach, based on *nonassociative geometry*, for describing both continuum and discrete spacetime. Among advanced models that propose discreteness, two that will be relevant to our work are Causal Sets (CS)[19–21] and Causal Dynamical Triangulations (CDT)[14, 22–28]. These approaches are based on the hypothesis that spacetime is discrete and causality is a fundamental principle.

Nonassociative geometry, being based on the theory of quasigroups and loops, provides a unified algebraic description of manifolds. In the case of smooth manifolds it is equivalent to conventional differential geometry [29, 30]. The main algebraic structures emerging in nonassociative geometry may be described as follows. Let $\mathfrak{M}$ be a manifold with an affine connection. Then, in a neighborhood of an arbitrary point $a \in \mathfrak{M}$ on one can introduce the geodesic local loop $\mathcal{Q}_a$ with a smooth partial ternary operation, $L_x^a y$, a being the neutral element of $\mathcal{Q}_a$, and $a, x, y \in \mathfrak{M}$. The operation $L_x^a y$ is uniquely defined through the parallel transport of the geodesic $\langle ay \rangle$, connecting

the point a with the point y, along the geodesic $\langle ax \rangle$. The family of local loops $\{\mathcal{Q}_a\}$ constructed in this way satisfies some natural algebraic identities and uniquely defines the space with affine connection. Next, one can introduce the left-invariant metric, $g^b(L_b^a x, L_b^a y) = g^a(x, y)$, enriching the manifold $\mathfrak{M}$ with a *metric structure*. In nonassociative geometry, the (nonlocal) equivalent of curvature is elementary holonomy, $h_{(x,y)}^a = L_a^y \circ L_y^x \circ L_x^a$.

In the present paper, we introduce a discrete model of emergent spacetime based on nonassociative geometry. In our approach events, being "atoms" of spacetime, are elements of a local loop with a partial operation. We assume that time is quantized and the distance between two neighboring atoms can not be less than the fundamental length, ℓ_p. The evolution of spacetime geometry is governed by a random/stochastic process conserving spacetime geometry. The result is a partially ordered set of events, and spacetime geometry is encoded in the nonassociative structure of spacetime. We apply our approach to study a toy model of $(2 + 1)$-D discrete spacetime and a discrete Friedmann-Robertson-Walker (FRW) cosmological model. We show that the evolution of the discrete spacetime corresponds to a radiation era in the continuous limit with the scale factor growing as $\sim \sqrt{t}$. The irreversibility of evolution is governed by a random process and by information loss as a consequence of the fundamental Planck length.

The paper is organized as follows. In section 2 we survey the foundations of the theory of smooth quasigroups and loops. In section 3, we give a brief introduction to nonassociative geometry, with detailed examples and applications to the de Sitter and Friedmann-Robertson-Walker (FRW) models of spacetime. In section 4 we introduce nonassociative discrete geometry. In section 5 we present a discrete model of spacetime based on nonassociative geometry with applications to $(2+1)$-D spacetime and the FRW model. In section 6 we discuss the results of numerical simulations and theoretical models for the implementation of our framework. In the **Supplemental Material**, we present technical details of our method and algorithms.

2. SMOOTH QUASIGROUPS AND LOOPS

During the last five decades, nonassociative structures have appeared in various fields of physics, and nowadays they play an important role in modern physics. Among others, one may mention the employment of octonions, Lie groupoids, and algebroids in the context of Yang-Mills theories and gauge theories on commutative but nonassociative fuzzy spaces [31–37], and attempts to create nonassociative quantum mechanics [38–42].

The nonassociative objects, such as 3-cocycles, emerge in quantum mechanics with the Dirac monopole and quantum field theory with chiral anomalies [43–55]. The chiral anomalies appear as the Schwinger terms in current algebra [53–56]. These terms, having a cohomological nature, are related to the failure of the Jacobi identity, $J(A, B, C) = 0$, where for a given triple of

operators (A, B, C) we define

$$J(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B]. \quad (1)$$

In particular, it has been shown that electric fields in chiral gauge theories satisfy the fourth-order identity, $J(E_i, E_j, [E_k, E_i]) = [J(E_i, E_j, E_k), E_i]$, instead of the Jacobi identity, and form the so-called Mal'cev algebra [53–57]. The Mal'cev algebra, being a tangent algebra of the analytic Moufang loop, generalizes the concept of Lie algebra, and for a given tangent Mal'cev algebra there exists a unique simply connected analytical Moufang loop.

Failure of the Jacobi identity implies violation of associativity and forces one to make use of nonassociative generalization of the Lie groups, such as smooth quasigroups and loops. The nonassociative algebraic structures, such as smooth quasigroups and loops, are of considerable potential interest for modern physics. Recently, the theory of smooth loops has been employed for description of Thomas precession, coherent states and geometric phases in quantum mechanics [58–61]. In general relativity, unlike special relativity, the Poincaré group does not exist in general spacetime. However, this problem can be overcome by introducing the Poincaré quasigroup [62, 63].

The theory of fiber bundles, being one of the important applications of Lie groups not only in mathematics but in physics as well, leads to gauge field theory. The nonassociative generalization of Lie groups allows us to construct nonassociative gauge theories with higher "nonassociative gauge symmetries" by using the theory of smooth loops [64–66]. In general terms, a consequence of nonassociativity is that the structure constants of the gauge algebra have to be changed by structure functions.

In what follows we outline the theory of smooth quasigroups and loops. While there is not a consensus about the birth date of the theory of smooth loops, one can consider the seminal work by Mal'cev [57] as a starting point for the development of this theory. General results on the subject, including reviews, may be found in [30, 67–69], Kikkawa [70], Belousov [71], Pflugfelder [72], and Batalin [73] (see also references therein). The point of departure is an identity of quasiassociativity leading to the infinitesimal theory of smooth loops.

Definition 2.1. Let $\langle Q, \cdot \rangle$ be a groupoid, i.e., a set with a binary operation $(a, b) \mapsto a \cdot b$. A groupoid $\langle Q, \cdot \rangle$ is called a *quasigroup* if each of the equations $a \cdot x = b$, and $y \cdot a = b$ has a unique solution, $x = a \backslash b$ and $y = b / a$, defining left ("\") and right ("/") division. A *loop* is a quasigroup with a two-sided identity, $a \cdot e = e \cdot a = a$, where e is a neutral element. A loop that is also a differentiable manifold and in which the operation $a \cdot b$ is a smooth map is called a *smooth loop*.

One can show that the following identities hold:

$$a \cdot (b \cdot c) = (a \cdot b) l_{(a,b)} c, \quad r_{(b,c)} a \cdot (b \cdot c) = (a \cdot b) c, \quad (2)$$

$$l_{(a,b)} = L_{a \cdot b}^{-1} \circ L_a \circ L_b, \quad r_{(b,c)} = R_{b \cdot c}^{-1} \circ R_c \circ R_b, \quad (3)$$

where $l_{(a,b)}$ is a *left associator*, $r_{(b,c)}$ is a *right associator*, L_a is a *left translation* and R_b is a *right translation*. The left and right translations are defined as follows: $L_a b = R_b a = a \cdot b$. As one can see, the loop becomes a group, if the associator is the identity

operator. Thus, the associator can be considered as a measure of nonassociativity of the operation in the loop.

Let $T_e(Q)$ be the tangent space at the neutral element $e \in Q$. Then, for each $X_e \in T_e(Q)$, one can define a smooth vector field on Q as

$$X_b = L_{b*}X_e, \quad b \in Q, \quad X_e \in T_e(Q), X_b \in T_b(Q),$$

where $L_{b*} : T_e(Q) \mapsto T_b(Q)$ is the differential of the left translation.

Definition 2.2. A vector field X on Q which satisfies the relation $L_{a*}X_b = \hat{l}_{(a,b)*}X_{a \cdot b}$ for any $a, b \in Q$ is called a *left fundamental* or *left quasi-invariant* vector field.

Let $\Gamma_i = R_i^j \partial / \partial a^j$, $i = 1, 2, \dots, r$, be a basis of left quasi-invariant vector fields. Then, for the commutator, $[\Gamma_i, \Gamma_j]$, we obtain

$$[\Gamma_i, \Gamma_j] = C_{ij}^p(a)\Gamma_p, \tag{4}$$

where $C_{ij}^p(a)$ are the *structure functions*. They satisfy the modified Jacobi identity,

$$C_{ij,n}^p R_k^n + C_{jk,n}^p R_i^n + C_{ki,n}^p R_j^n + C_{ij}^l C_{kl}^p + C_{jk}^l C_{il}^p + C_{ki}^l C_{jl}^p = 0. \tag{5}$$

Comment 2.1. The infinitesimal theory of Lie groups is based on the associativity of the operation, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. In a similar way, the quasiassociativity identity,

$$L_a \circ L_b c = L_{(a \cdot b)} \circ l_{(a,b)} c, \tag{6}$$

yield the infinitesimal theory of smooth loops [66].

2.1. Examples

Example 2.1. **Loop $\langle Q(\mathbb{R}/\mathbb{Z}), * \rangle$.** The analytical loop $Q(\mathbb{R}/\mathbb{Z})$ is a loop with the binary operation given by [74, 75]

$$x * y = x + y + f(x) + f(y) - f(x + y), \tag{7}$$

where $f(x) = (1 - \cos 2\pi x)/4$.

Example 2.2. **Loop $Q\mathbb{C}$.** Let $\mathbb{C}$ be the complex plane. It becomes the loop $Q\mathbb{C}$, if we define the following nonassociative operation:

$$R_\eta \zeta = L_\zeta \eta = \zeta \cdot \eta = \frac{\zeta + \eta}{1 - \bar{\zeta}\eta}, \quad \zeta, \eta \in \mathbb{C}, \tag{8}$$

$$L_\zeta^{-1} \eta = \frac{\eta - \zeta}{1 + \bar{\zeta}\eta}, \quad \zeta, \eta \in Q\mathbb{C}, \tag{9}$$

where the bar denotes complex conjugation. The associator $l_{(\zeta, \eta)}$ is found to be

$$l_{(\zeta, \eta)} \xi = \frac{1 - \zeta\bar{\eta}}{1 - \eta\bar{\zeta}} \xi. \tag{10}$$

The basis of left quasi-invariant vectors and 1-forms is given by

$$\Gamma_1 = (1 + |\eta|^2)\partial_\eta, \quad \Gamma_2 = (1 + |\eta|^2)\partial_{\bar{\eta}} \tag{11}$$

$$\theta^1 = \frac{d\eta}{1 + |\eta|^2}, \quad \theta^2 = \frac{d\bar{\eta}}{1 + |\eta|^2} \tag{12}$$

The computation of the commutator results in:

$$[\Gamma_1, \Gamma_2] = \bar{\eta}\Gamma_2 - \eta\Gamma_1, \tag{13}$$

and for the structure functions we find that $C_{12}^1 = -\eta$, $C_{12}^2 = \bar{\eta}$.

One can show that the metric based on the left fundamental basis forms, $ds^2 = \theta^1 \theta^2$, is a left invariant metric. As one can see, this is also the Fubini-Studi metric:

$$ds^2 = \frac{d\eta \, d\bar{\eta}}{(1 + |\eta|^2)^2}. \tag{14}$$

Comment 2.2. The loop $Q\mathbb{C}$ is locally isomorphic to the Bol loop QS^2 [16, 30]. The latter is related to the two-sphere S^2, which admits a natural quasigroup structure induced by the operation (8). The isomorphism is established by using the stereographic projection from the south pole of the sphere: $\zeta = e^{i\varphi} \tan(\theta/2)$.

Example 2.3. **Loop $QSU(2)$.** Consider a set of the unitary matrices $QSU(2) \subset SU(2)$. An arbitrary element $U_\eta \in QSU(2)$ is given by

$$U_\eta = \frac{1}{\sqrt{1 + |\eta|^2}} \begin{pmatrix} 1 & \eta \\ -\bar{\eta} & 1 \end{pmatrix}, \quad U_\eta \in QSU(2). \tag{15}$$

We introduce the binary (nonassociative) operation on $QSU(2)$ as follows:

$$U_\eta * U_\zeta = U_\eta U_\zeta \Lambda(\eta, \zeta), \tag{16}$$

where

$$\Lambda(\eta, \zeta) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad \varphi = 2\arg(1 - \bar{\eta}\zeta).$$

The matrix loop $QSU(2)$ forms the nonassociative representation of the loop $Q\mathbb{C} \simeq QS^2$. Indeed, writing the result of multiplication of two elements, $U_\eta, U_\zeta \in QSU(2)$, as

$$U_\eta * U_\zeta = \frac{1}{\sqrt{1 + |\eta \cdot \zeta|^2}} \begin{pmatrix} 1 & \eta \cdot \zeta \\ -\overline{\eta \cdot \zeta} & 1 \end{pmatrix}, \tag{17}$$

we find (see example 2.2)

$$L_\eta \zeta = \frac{\eta + \zeta}{1 - \bar{\eta}\zeta}, \quad l_{(\eta, \zeta)}\xi = \frac{1 - \eta\bar{\zeta}}{1 - \bar{\eta}\zeta}\xi, \quad \zeta, \xi, \eta \in \mathbb{C}. \tag{18}$$

Example 2.4. **Loop QH^2.** Let $D \subset \mathbb{C}$ be the open unit disk, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Then, the set of complex numbers with the operation [58, 59, 64, 65]

$$\zeta * \eta = \frac{\zeta + \eta}{1 + \bar{\zeta}\eta}, \quad \zeta, \eta \in D. \tag{19}$$

forms the two-sided loop QH^2. The associator, $l_{(\zeta, \eta)}$, on QH^2 is defined by

$$l_{(\zeta, \eta)}\xi = \frac{1 + \zeta\bar{\eta}}{1 + \eta\bar{\zeta}}\xi. \tag{20}$$

The basis of left quasi-invariant vectors and 1-forms is given by

$$\Gamma_1 = (1 - |\eta|^2)\partial_\eta, \quad \Gamma_2 = (1 - |\eta|^2)\partial_{\bar\eta} \quad (21)$$

$$\theta^1 = \frac{d\eta}{1 - |\eta|^2}, \quad \theta^2 = \frac{d\bar\eta}{1 - |\eta|^2}. \quad (22)$$

The computation of the commutator yields

$$[\Gamma_1, \Gamma_2] = \eta\Gamma_1 - \bar\eta\Gamma_2. \quad (23)$$

Using the left fundamental basis forms, we obtain the hyperboloic (left invariant) metric

$$ds^2 = \theta^1\theta^2 = \frac{d\eta\, d\bar\eta}{(1 - |\eta|^2)^2}. \quad (24)$$

The loop QH^2 is isomorphic to the geodesic loop of two-dimensional Lobachevsky space. The isomorphism is established by using the relation $\zeta = e^{i\varphi}\tanh(\theta/2)$, where (θ, φ) are inner coordinates on the upper part of two-sheeted hyperboloid, H^2.

Example 2.5. **Loop** $QSU(1, 1)$. Consider a set of the unitary matrices $QSU(1, 1) \subset SU(1, 1)$. An arbitrary element, $U_\eta \in QSU(1, 1)$, is given by

$$U_\eta = \frac{1}{\sqrt{1 - |\eta|^2}}\begin{pmatrix} 1 & \eta \\ \bar\eta & 1 \end{pmatrix}, \quad \eta \in D, \quad U_\eta \in QSU(1, 1), \quad (25)$$

where $D \subset \mathbb{C}$ is the open unit disk, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. We introduce the nonassociative operation as follows:

$$U_\eta * U_\zeta = U_\eta U_\zeta \Lambda(\eta, \zeta), \quad (26)$$

where

$$\Lambda(\eta, \zeta) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad \varphi = 2\arg(1 + \bar\eta\zeta). $$

The loop $QSU(1, 1)$ forms the nonassociative representation of the loop QH^2. Indeed, writing Equation (26) as

$$U_\eta * U_\zeta = \frac{1}{\sqrt{1 - |\eta\cdot\zeta|^2}}\begin{pmatrix} 1 & \eta\cdot\zeta \\ \overline{\eta\cdot\zeta} & 1 \end{pmatrix}, \quad (27)$$

we find (see Example 2.4)

$$L_\eta\zeta \equiv \eta\cdot\zeta = \frac{\eta + \zeta}{1 + \bar\eta\zeta}, \quad l_{(\eta,\zeta)}\xi = \frac{1 + \eta\bar\zeta}{1 + \bar\eta\zeta}\xi, \quad \zeta, \xi, \eta \in D. \quad (28)$$

Example 2.6. **Loop** $QH_\mathbb{R}$ (*De Sitter loop*). Let us consider the algebra of quaternions, $H_\mathbb{C}$, over a complex field $\mathbb{C}(1, i)$,

$$H_\mathbb{C} = \{\alpha + \beta i + \gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{C}\}. \quad (29)$$

The multiplication of quaternions is defined by the following rules for (i, j, k):

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, ki = -ik = j, \quad ij = -ji = k. \quad (30)$$

For a given quaternion, $q = \alpha + \beta i + \gamma j + \delta k$, the quaternionic conjugation is defined as

$$q^+ = \alpha - \beta i - \gamma j - \delta k. \quad (31)$$

Further, we restrict ourselves to consideration of the set of quaternions $H_\mathbb{R} \subset H_\mathcal{C}$:

$$H_\mathbb{R} = \{\zeta = \zeta^0 + i(\zeta^1 i + \zeta^2 j + \zeta^3 k) : i^2 = -1, \ i \in \mathbb{C},$$
$$\zeta^0, \zeta^1, \zeta^2, \zeta^3 \in \mathbb{R}\}, \quad (32)$$

with the norm defined as $\|\zeta\|^2 = \zeta\zeta^+ = (\zeta^0)^2 - (\zeta^1)^2 - (\zeta^3)^2 - (\zeta^4)^2$.

We introduce the binary operation

$$\zeta * \eta = (\zeta + \eta)\Big/\Big(1 + \frac{K}{4}\zeta^+\eta\Big), \quad \zeta, \eta \in H_\mathbb{R}, \quad (33)$$

where $/$ denotes right division and K is constant. The set of quaternions $H_\mathbb{R}$ with the binary operation (77) forms the loop $QH_\mathbb{R}$. The associator is given by

$$l_{(\zeta,\eta)}\xi = \Big(1 + \frac{K}{4}\zeta\eta^+\Big)\xi\Big/\Big(1 + \frac{K}{4}\zeta^+\eta\Big). \quad (34)$$

Comment 2.3. For $K > 0$ the loop $QH_\mathbb{R}$ is isomorphic to the geodesic loop of spacetime of constant positive curvature (*de Sitter space*) [16].

Example 2.7. **Loop** $QH_\mathbb{R}^\mathcal{F}$. Consider the set of quaternions:

$$H_\mathbb{R}^\mathcal{F} = \{x = \zeta^0 + a(\zeta^0)\boldsymbol\zeta, \boldsymbol\zeta = i(\zeta^1 i + \zeta^2 j + \zeta^3 k)\}, \quad (35)$$

where $a(\zeta^0)$ is a smooth function. We introduce a binary operation $L_x y = x * y$, $x, y \in H_\mathbb{R}^\mathcal{F}$ as

$$L_x y = \zeta^0 + \eta^0 + a(\zeta^0 + \eta^0)(\boldsymbol\zeta + \boldsymbol\eta)\Big/\Big(1 + \frac{K}{4}\boldsymbol\zeta^+\boldsymbol\eta\Big), \quad (36)$$

$$L_x^{-1}y = \eta^0 - \zeta^0 + a(\eta^0 - \zeta^0)(\boldsymbol\eta - \boldsymbol\zeta)\Big/\Big(1 - \frac{K}{4}\boldsymbol\zeta^+\boldsymbol\eta\Big), \quad (37)$$

where $x = \zeta^0 + a(\zeta^0)\boldsymbol\zeta$, $y = \eta^0 + a(\eta^0)\boldsymbol\eta$, and $/$ denotes the right division. It is easy to show that $e = 0$ is the neutral element, so that $L_e x = x$. The associator is defined by

$$l_{(x,y)}z = \xi^0 + a(\xi^0)\Big(1 + \frac{K}{4}\boldsymbol\eta^+\boldsymbol\zeta\Big)\boldsymbol\xi\Big/\Big(1 + \frac{K}{4}\boldsymbol\zeta^+\boldsymbol\eta\Big), \quad (38)$$

where we set $z = \xi^0 + a(\xi^0)\boldsymbol\xi$. The set of quaternions, $H_\mathbb{R}^\mathcal{F}$ with the binary operation $*$, forms a loop, which will be denoted as $QH_\mathbb{R}^\mathcal{F}$.

Example 2.8. **Moufang loop**. A loop Q is a Moufang loop if, for all $a, b, c \in Q$, the following identities hold:

$$c(a(cb)) = ((ca)c)b, \quad a(c(bc)) = ((ac)b)c, \quad (39)$$
$$(ca)(bc) = (c(ab))c, \quad (ca)(bc) = c((ab)c). \quad (40)$$

In particular, the sphere S^7, identified with the unit octonions, may be considered a smooth Moufang loop with the binary operation defined by the octonionic algebra. Other examples of the analytical Moufang loops associated with octonions are the spaces $S^3 \times R^4$ and $S^7 \times R^7$.

3. NONASSOCIATIVE GEOMETRY IN BRIEF

In this section we outline the algebraic foundations of *nonassociative geometry* (For details see [29, 30, 76–78]).

Definition 3.1. Let $\langle \mathfrak{M}, \cdot, e \rangle$ be a partial *groupoid* with a binary operation $(x, y) \mapsto x \cdot y$ and a neutral element e, $x \cdot e = e \cdot x = x$. Consider $\mathfrak{M}$ to be a smooth manifold (at least C^1-smooth) and the operation of multiplication (at least C^1-smooth) to be defined in some neighborhood U_e. Then, $\langle \mathfrak{M}, \cdot, e \rangle$ is called a *partial loop on* $\mathfrak{M}$.

Remark 3.1. The operation of multiplication is locally invertible. This implies that in some neighborhood of the neutral element e one has

$$L_a(L_a^{-1} x) = x, \quad R_a(R_a^{-1} x) = x,$$

where $x \cdot y = L_x y = R_y x$,

The vector fields A_j and B_j defined on U_e as

$$A_j(x) = ((L_x)_{*,e})^i_j \frac{\partial}{\partial x^i} = L^i_j(x) \frac{\partial}{\partial x^i}, \quad (41)$$

$$B_j(x) = ((R_x)_{*,e})^i_j \frac{\partial}{\partial x^i} = R^i_j(x) \frac{\partial}{\partial x^i} \quad (42)$$

are called the *left basic fundamental fields* and *right basic fundamental fields*, respectively.

The solution of the equation

$$\frac{df^i(t)}{dt} = L^i_j(f(t)) X^j, \quad f(0) = e, \quad (43)$$

can be written as $f(t) = \mathrm{Exp}\, tX$ and defines the *exponential map*,

$$\mathrm{Exp} : X \in T_e(\mathfrak{M}) \longrightarrow \mathrm{Exp} X \in \mathfrak{M}.$$

Employing the exponential map, one can define a new operation, $tx = \mathrm{Exp}(t\mathrm{Exp}^{-1} x)$, called the *left canonical unary operation*.

Definition 3.2. A smooth loop $\langle \mathfrak{M}, \cdot, e \rangle$ equipped with its canonical left unary operations is called the *left canonical preodule* $\langle \mathfrak{M}, \cdot, (t)_{t \in \mathbb{R}}, e \rangle$.

Introducing one more operation,

$$x + y = \mathrm{Exp}(\mathrm{Exp}^{-1} x + \mathrm{Exp}^{-1} y), \quad (44)$$

we obtain the *canonical left prediodule* of a loop, denoted as $\langle \mathfrak{M}, \cdot, +, (t)_{t \in \mathbb{R}}, e \rangle$.

Definition 3.3. A canonical left preodule (prediodule) is called the *left odule (diodule)* if the *monoassociativity* property, $tx \cdot ux = (t + u)x$ is satisfied.

Definition 3.4. Let $\mathfrak{M}$ be a smooth manifold and $L : (x, y, z) \in \mathfrak{M} \mapsto L(x, y, z) \in \mathfrak{M}$ be a smooth partial ternary operation, such that $x_{\dot{a}} y = L(x, a, z)$ defines in some neighborhood of the point a the loop with the neutral a. Then, the pair $\langle \mathfrak{M}, L \rangle$ is called a *loopuscular structure (manifold)*.

Definition 3.5. A smooth manifold $\mathfrak{M}$ with a smooth partial ternary operation L and smooth binary operations $\omega_t : (a, b) \in \mathfrak{M} \times \mathfrak{M} \mapsto \omega_t(a, b) = t_a b \in \mathfrak{M}$, $(t \in \mathbb{R})$, such that $x_{\dot{a}} y = L(x, a, y)$ and $t_a z = \omega_t(a, z)$ determine in some neighborhood of an arbitrary point a the odule with the neutral element a, is called a *left odular structure (manifold)* $\langle \mathfrak{M}, L, (\omega_t)_{t \in \mathbb{R}} \rangle$.

Let $\langle \mathfrak{M}, L, (\omega_t)_{t \in \mathbb{R}} \rangle$ and $\langle \mathfrak{M}, N, (\omega_t)_{t \in \mathbb{R}} \rangle$ be odular structures. Then, we define a *diodular structure (manifold)* as $\langle \mathfrak{M}, L, N, (\omega_t)_{t \in \mathbb{R}} \rangle$. If, in addition, $x_{\underset{a}{+}} y = N(x, a, y)$ and $t_a x = \omega_t(a, x)$ define a vector space, the diodular structure is called a *linear diodular structure*.

A diodular structure is said to be *geodiodular* if the following identities hold:

$$L^{t_a x}_{u_a x} \circ L^a_{t_a x} = L^a_{u_a x} \quad (L^a_x y = L(x, a, y)), \quad \text{(the first geoodular identity)} \quad (45)$$

$$L^a_x \circ t_a = t_x \circ L^a_x, \quad \text{(the second geoodular identity)} \quad (46)$$

$$L^a_x N(y, a, z) = N(L^a_x y, x, L^a_x z) \quad \text{(the third geoodular identity).} \quad (47)$$

Definition 3.6. Let $\mathfrak{M}$ be a C^k-smooth $(k \geq 3)$ affinely connected manifold, with the following operations given on $\mathfrak{M}$ (locally):

$$L^a_x y = x_{\dot{a}} y = \mathrm{Exp}_x \tau^a_x \mathrm{Exp}^{-1}_a y, \quad (48)$$

$$\omega_t(a, z) = t_a z = \mathrm{Exp}_a t \mathrm{Exp}^{-1}_a z, \quad (49)$$

$$N(x, a, y) = x_{\underset{a}{+}} y = \mathrm{Exp}_a(\mathrm{Exp}^{-1}_a x + \mathrm{Exp}^{-1}_a y), \quad (50)$$

where Exp_x is the exponential map at the point x, and τ^a_x is the parallel transport along the geodesic connecting the points a and x. This construction equips a manifold $\mathfrak{M}$ with the linear geodiodular structure, called a *natural linear geodiodular structure of an affinely connected manifold* $(\mathfrak{M}, \nabla)$. In **Figure 1**, the main operation (48) – (50) characterizing nonassociative geometry are illustrated.

Remark 3.2. Any C^k-smooth $(k \geq 3)$ manifold with affine connection can be considered as a geodiodular structure. A geodiodular space $\mathcal{M}^+_a = \langle \mathfrak{M}, \underset{a}{+}, a, (t_a)_{t \in \mathbb{R}} \rangle$ plays the role of a tangent space.

Definition 3.7. Let $\langle \mathfrak{M}, L \rangle$ be a loopuscular structure of a smooth manifold $\mathfrak{M}$. Then the formula

$$\nabla_{X_a} Y = \left\{ \frac{d}{dt} \left([(L^a_{g(t)})_{*,a}]^{-1} Y_{g(t)} \right) \right\}_{t=0}, \quad (51)$$

$$g(0) = a, \quad \dot{g}(0) = X_a, \quad (52)$$

Y being a vector field in the neighborhood of a point a, defines the *tangent affine connection*. In coordinates, the components of the affine connection are

$$\Gamma^i_{jk}(a) = - \left[\frac{\partial^2 (x_{\dot{a}} y)^i}{\partial x^j \partial y^k} \right]_{x=y=a}. \quad (53)$$
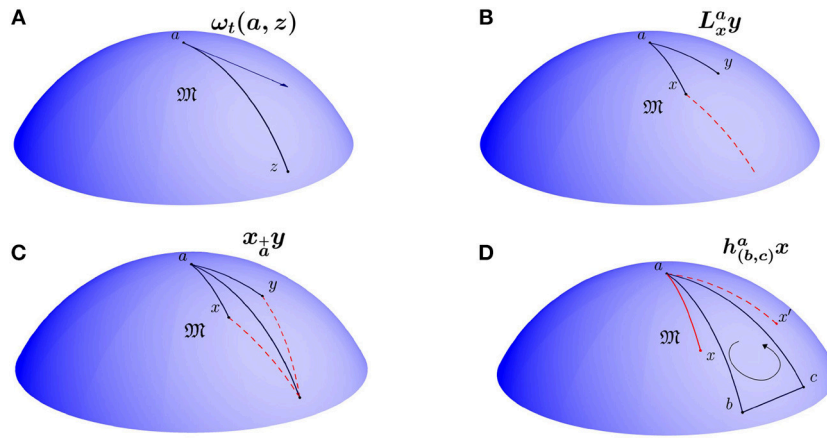
FIGURE 1 | Main operations describing nonassociative geometry: **(A)** Multiplication of a "vector" (geodesic) by a scalar (parameter t); **(B)** Parallel translation of the geodesic (ay) along the geodesic (ax); **(C)** Addition of "vectors" (geodesics on the manifold $\mathfrak{M}$); **(D)** Parallel translation of the geodesic (ax) along the geodesic triangle (abc) .

Definition 3.8. Let $\langle \mathfrak{M}, L \rangle$ be a loopuscular structure. Then,

$$h_{(b,c)}^{a} = (L_c^a)^{-1} \circ L_c^b \circ L_b^a \qquad (54)$$

is called the *elementary holonomy*.

The elementary holonomy describes the parallel transport along a geodesic triangle path (see **Figure 1**). Thus, it can be considered as some integral curvature. In the smooth case, differentiating $(h_{(x,y)}^a)^i$ at the point $a \in \mathfrak{M}$, we obtain (up to a numerical factor) the curvature tensor as

$$R^i{}_{jkl}(a) = 2 \left[\frac{\partial^3 (h_{(x,y)}^a z)^i}{\partial x^l \partial y^k \partial z^j} \right]_{x=y=z=a}. \qquad (55)$$

Remark 3.3. The curvature of a nonassociative space implies non-trivial elementary holonomy, $h^a(x, y) \neq \mathrm{id}$. If $h^a(x, y) = \mathrm{id}$ for any $a, x, y \in \mathfrak{M}$, we obtain a space with absolute parallelism. If in addition, the relationship $x +_a y = x_a y$ holds, that means the absence of torsion, and the space $\mathfrak{M}$ becomes a classical affine space (see chapter 3 in [30]).

Employing (54), one can show that the elementary holonomy satisfies the so-called *odular Bianchi identities*:

$$h_{(z,x)}^a \circ h_{(y,z)}^a \circ h_{(x,y)}^a = (L_x^a)^{-1} \circ h_{(y,z)}^x \circ L_x^a. \qquad (56)$$

These identities can be considered as a non-local form of the usual Bianchi identities. In the linear approximation, from (56) we obtain the Bianchi identities in their conventional form.

3.1. Examples

Example 3.1. **Two-dimensional sphere S_R^2.** The two-sphere S_R^2 with a radius R can be equipped by a natural loop structure, that may be described as follows [58, 59, 64, 65]. Let $\mathbb{C}$ be a complex plane and $\zeta, \eta \in \mathbb{C}$. Let us consider the loop formed by the

complex numbers with the non-associative multiplication (left translation) given by

$$L_\zeta \eta = \frac{\zeta + \eta}{1 - \overline{\zeta} \eta / R^2}, \quad \zeta, \eta \in \mathbb{C}. \qquad (57)$$

This loop is isomorphic to the local loop related to S_R^2. The isomorphism between points of the sphere, S_R^2, and of the complex plane $\mathbb{C}$ can be established by using the stereographic projection from the south pole of the sphere, writing $\zeta = R \tan(\theta/2) e^{i\varphi}$.

The computation of the associator yields

$$l_{(\zeta,\eta)}\xi = \frac{1 - \zeta \overline{\eta}/R^2}{1 - \eta \overline{\zeta}/R^2} \xi. \qquad (58)$$

One can show that for the sphere the elementary holonomy is determined by the associator, $h_{(\zeta,\eta)} = l_{(\zeta, L_\zeta^{-1}\eta)}$ [30, 76]. We obtain

$$h_{(\zeta,\eta)}\xi = \frac{1 + \overline{\zeta}\eta/R^2}{1 + \zeta\overline{\eta}/R^2} \xi. \qquad (59)$$

The left invariant metric on S_R^2 can be defined as

$$g(\zeta, \xi) = \frac{4|\xi - \zeta|^2}{(1 + |\zeta|^2/R^2)(1 + |\xi|^2/R^2)}. \qquad (60)$$

The geodesic distance between two points on S_R^2 is related to the left invariant metric as follows:

$$\cos \frac{d_{12}}{R} = 1 - \frac{\ell_{12}^2}{2R^2}, \qquad (61)$$

where

$$\ell_{12} = \frac{2|\xi_1 - \xi_2|}{\sqrt{(1 + |\xi_1|^2/R^2)(1 + |\xi_2|^2/R^2)}}. \qquad (62)$$

For $d_{12}/R \ll 1$ we obtain $d_{12} \approx \ell_{12}$. Using spherical coordinates, one can show that (61) can be recast as the cosine rule in spherical trigonometry,

$$\cos \theta_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \varphi_{12}, \qquad (63)$$

where $\theta_{12} = d_{12}/R$, $\theta_1 = 2 \tan^{-1} |\xi_1|$, $\theta_2 = 2 \tan^{-1} |\xi_2|$ and $\varphi_{12} = \varphi_1 - \varphi_2$.

Let $\xi = \zeta + d\zeta$. Then, (60) leads to

$$g(d\zeta, d\zeta) = \frac{4 d\zeta \, d\bar{\zeta}}{(1 + |\zeta|^2/R^2)^2},$$

the well-known expression for the element of length of S_R^2. Computation of the curvature tensor yields

$$R^{\zeta}{}_{\zeta\zeta\bar{\zeta}} = \frac{2}{R^2 \left(1 + |\zeta|^2/R^2\right)^2}, \qquad (64)$$

and using Equation (59) we obtain

$$R^{\zeta}{}_{\bar{\zeta}\zeta\zeta}(0) = 2 \left[\frac{\partial^3 (h_{(\zeta,\eta)}\xi)}{\partial \zeta \, \partial \bar{\eta} \, \partial \xi} \right]_{\zeta=\eta=\xi=0},$$

which is consistent with Equation (55).

Example 3.2. **Lobachevsky space**. Using the isomorphism established in Example 2.4 between the loop QH^2 and Lobachevsky space, one can consider the upper part the two-sheeted unit hyperboloid, H^2, as a nonassociative space with the left translations defined by

$$L_\zeta \eta = \frac{\zeta + \eta}{1 + \bar{\zeta}\eta}, \qquad \zeta, \eta \in D. \qquad (65)$$

In terms of the inner coordinates (θ, φ) on H^2, an arbitrary complex number $\zeta \in D$ can be written as $\zeta = e^{i\varphi} \tanh(\theta/2)$.

The computation of the elementary holonomy yields

$$h_{(\zeta,\eta)}\xi = \frac{1 - \bar{\zeta}\eta}{1 - \zeta\bar{\eta}}\xi. \qquad (66)$$

We introduce the left invariant metric on H^2 as

$$g(\zeta, \xi) = \frac{4|\xi - \zeta|}{\sqrt{(1 - |\zeta|^2)(1 - |\xi|^2)}}. \qquad (67)$$

The Poincaré disk model is associated to the hyperbolic metric $d(\zeta, \xi)$, assigning to each pair of points $\zeta, \xi \in D$ the distance [79]

$$\cosh d(\zeta, \xi) = 1 + \frac{2|\xi - \zeta|^2}{(1 - |\zeta|^2)(1 - |\xi|^2)}. \qquad (68)$$

Thus, the hyperbolic distance between two points on H^2 is related to the left invariant metric on D [see Equation (67)] as follows: $\cos d = 1 + \ell^2/2$, where we set $\ell = g(\zeta, \xi)$. For $d \ll 1$ we obtain $d \approx \ell$.

Example 3.3. **De Sitter spacetime.** De Sitter space, being a spacetime of constant positive curvature with topology $S^3 \times \mathbb{R}$, can be treated as a hyperboloid embedded in five-dimensional Minkowski space [80, 81],

$$Z_a Z^a - Z_0 Z^0 = -1/K, \ a = 1, \dots, 4, \ K > 0, \qquad (69)$$

with metric given by

$$ds^2 = (dZ^0)^2 - dZ_a dZ^a. \qquad (70)$$

The induced metric of the de Sitter space can be written as

$$ds^2 = \frac{dt^2 - dx^2 - dy^2 - dz^2}{\left(1 - \frac{K}{4}(t^2 - x^2 - y^2 - z^2)\right)^2}. \qquad (71)$$

The de Sitter loop, $QH_{\mathbb{R}}$, introduced in section 2, is isomorphic to the geodesic loop of de Sitter space. It has a natural geodiodular structure induced by the quaternionic algebra. The left invariant metric on $QH_{\mathbb{R}}$ can be defined as follows [16]:

$$g^\zeta(\xi, \xi) = \frac{\|\xi - \zeta\|^2}{\|1 - \frac{K}{4}\bar{\zeta} + \xi\|^2}. \qquad (72)$$

Letting $\xi = \zeta + d\zeta$, this leads to the de Sitter metric (71):

$$g(d\zeta, d\zeta) = \frac{(d\zeta^0)^2 - (d\zeta^1)^2 - (d\zeta^2)^2 - (d\zeta^3)^2}{\left(1 - \frac{K}{4}((\zeta^0)^2 - (\zeta^1)^2 - (\zeta^2)^2 - (\zeta^3)^2)\right)^2}.$$

For symmetric spaces the elementary holonomy is determined by the associator, $h_{(\zeta,\eta)}\xi = l_{(\zeta, L_\zeta^{-1}\eta)}\xi$ [30, 76]. Taking this into account, we obtain

$$h_{(\zeta,\eta)}\xi = (1 - \frac{K}{4}\eta\zeta^+)\xi/(1 - \frac{K}{4}\eta^+\zeta). \qquad (73)$$

Applying (55), we obtain the curvature tensor of de Sitter space:

$$R_{\mu\nu\lambda\sigma} = -\frac{K}{2}\varepsilon_{\mu\nu\kappa\delta}\varepsilon^{\kappa\delta}{}_{\lambda\sigma}. \qquad (74)$$

Comment 3.1. De Sitter space can be obtained as a solution of the diodular Einstein's equations with a "Λ-term" [15].

Example 3.4. **Friedmann-Robertson-Walker spacetime.** The metric of the general Friedmann-Robertson-Walker (FRW) model may be written as [80, 81]

$$ds^2 = d\tau^2 - a^2(\tau)\left(dr^2 + \Sigma^2(r, K)\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right), \qquad (75)$$

where $\Sigma(r, K) = \sin r, \ r, \ \text{or} \ \sinh r$, respectively, if $K = 1, 0$, or -1. It can be transformed to the form

$$ds^2 = d\tau^2 - a^2(\tau)\frac{(d\zeta^1)^2 + (d\zeta^2)^2 + (d\zeta^3)^2}{\left(1 + \frac{K}{4}((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2)\right)^2}, \qquad (76)$$

which will be used below.

We introduce a binary operation in the neighborhood of the point $\tau \in H_{\mathbb{R}}^{\mathcal{F}}$ as

$$
\begin{aligned}
L_x y &= x * y \\
&= \zeta^0 + \eta^0 - \tau \\
&\quad + a(\zeta^0 + \eta^0 - \tau)(\zeta + \eta) \Big/ \Big(1 + \frac{K}{4} \zeta^+ \eta\Big), \quad x, y \in \mathsf{H}_{\mathbb{R}}^{\mathcal{F}},
\end{aligned}
\tag{77}
$$

where / denotes right division (see Example 2.7), $x = \zeta^0 + a(\zeta^0)\zeta$ and $y = \eta^0 + a(\eta^0)\eta$, with $e = \tau$ being the neutral element. The inverse operation is found to be

$$
L_x^{-1} y = \tau + \eta^0 - \zeta^0 + a(\tau + \eta^0 - \zeta^0)(\eta - \zeta) \Big/ \Big(1 - \frac{K}{4} \zeta^+ \eta\Big). \tag{78}
$$

The associator is given by

$$
l_{(x,y)} z = \xi^0 + a(\xi^0)\Big(1 + \frac{K}{4}\eta^+\zeta\Big)\xi \Big/ \Big(1 + \frac{K}{4}\zeta^+\eta\Big), \tag{79}
$$

where $z = \xi^0 + a(\xi^0)\xi$. The set of quaternions, $\mathsf{H}_{\mathbb{R}}^{\mathcal{F}}$, with the binary operation $*$ forms a loop, that will be denoted as $\mathsf{QH}_{\mathbb{R}}^{\mathcal{F}}$. It is easy to show that the quaternionic algebra induces a natural geoodular structure on $\mathsf{QH}_{\mathbb{R}}^{\mathcal{F}}$.

Finally, we define the left invariant metric on $\mathsf{QH}_{\mathbb{R}}^{\mathcal{F}}$ as

$$
g(\xi, \zeta) = \big(\zeta^0 - \xi^0\big)^2 - \frac{a^2(\tau + \zeta^0 - \xi^0)\|\boldsymbol{\xi} - \boldsymbol{\zeta}\|^2}{\big\|1 - \frac{K}{4}\boldsymbol{\xi}^+\boldsymbol{\xi}\big\|\big\|1 - \frac{K}{4}\boldsymbol{\zeta}^+\boldsymbol{\zeta}\big\|}, \tag{80}
$$

where $\zeta = \zeta^0 + a(\zeta^0)\boldsymbol{\xi}$ and $\xi = \xi^0 + a(\xi^0)\boldsymbol{\xi}$. If we let $\boldsymbol{\xi} = \boldsymbol{\zeta} + d\boldsymbol{\zeta}$ and $\xi^0 = \xi^0 + d\tau$, then (80) yields the FRW metric:

$$
ds^2 = d\tau^2 - a^2(\tau)\frac{(d\zeta^1)^2 + (d\zeta^2)^2 + (d\zeta^3)^2}{\big(1 + \frac{K}{4}((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2)\big)^2}.
$$

4. NONASSOCIATIVE DISCRETE GEOMETRY

Since in case of discrete spaces we lack a smooth structure, we must use only algebraic structures to study them. Thus, it is quite natural to employ nonassociative geometry as the adequate algebraic approach (for details see [15–18]).

Consider a nonassociative finite space $\mathcal{M} = \langle \mathfrak{M}, L, N, (\omega_t)_{t \in \mathbb{R}} \rangle$ equipped with (partial) operations: $L(x, y, z) = x \underset{y}{\cdot} z$, $N(x, y, z) = x \underset{y}{+} z$ and $\omega_t(x, y) = t_x y$ ($t \in \mathbb{R}$). This means that

- $\langle \mathfrak{M}, \underset{a}{\cdot}, (t_a)_{t \in \mathbb{R}} \rangle$ is an odule with a neutral $a \in M$, for any $a \in \mathfrak{M}$,
- $\langle \mathfrak{M}, \underset{a}{+}, (t_a)_{t \in \mathbb{R}} \rangle$ is a n-dimensional vector space (with zero element $a \in \mathfrak{M}$),

- The following geoodular identities are valid:

$$
L_{u_a^a x}^{t_a x} \circ L_{t_a x}^a = L_{u_a^a x}^a \quad (L_x^a y = L(x, a, y)), \quad \text{(the first geoodular identity)} \tag{81}
$$

$$
L_x^a \circ t_a = t_x \circ L_x^a, \quad \text{(the second geoodular identity)} \tag{82}
$$

$$
L_x^a N(y, a, z) = N(L_x^a y, x, L_x^a z) \quad \text{(the third geoodular identity)} \tag{83}
$$

A curve in discrete space can be introduced as an ordered set of points, $\gamma = (a_1, a_2, \cdots a_s)$ and, further, we can define the vector y (of $\mathcal{M}_{a_s}^+$) parallel to a vector x (of $\mathcal{M}_{a_1}^+$) along this curve γ as

$$
y = L_{a_s}^{a_{s-1}} \cdots L_{a_3}^{a_2} L_{a_2}^{a_1} x \quad (L_b^a x \overset{\text{def}}{=} b \underset{a}{\cdot} x), \tag{84}
$$

where $\mathcal{M}_a^+ = \langle \mathfrak{M}, \underset{a}{+}, a, (t_a)_{t \in \mathbb{R}} \rangle$. The curvature of a discrete space manifests as a non-trivial elementary holonomy, $h^a(x, y) \neq \mathrm{id}$,

$$
h^a(x, y) = L_a^y \circ L_y^x \circ L_x^a. \tag{85}
$$

Given a nonassociative space (in other words, a finite affinely connected space) one can enrich it by adding a metric structure. Specifically, we can define additionally the left invariant metric, $g_b(L_b^a x, L_b^a y) = g_a(x, y)$[17].

In what follows we consider some examples of nonassociative discrete spaces, including de Sitter space and FRW spacetime.

4.1. Lobachevsky Space

Here we consider a simple discrete model based on the loop QH^2. Let $\mathfrak{H}_n \subset D$ be a triangulated discrete topological space, with n being the number of vertices. Then $\mathfrak{H}_n$ becomes a nonassociative space if we define the partial nonassociative operation preserving the geometry of finite space, $\mathfrak{H}_n$,

$$
\zeta_{pq}^n = \frac{\zeta_p^n + \zeta_q^n}{1 + \bar{\zeta}_p^n \zeta_q^n}, \quad \zeta_p^n, \zeta_q^n, \zeta_{pq}^n \in \mathfrak{H}_n \subset D. \tag{86}
$$

The "metric" and elementary holonomy are defined by

$$
g(\zeta_p^n, \zeta_q^n) = \frac{4|\zeta_p^n - \zeta_q^n|^2}{(1 - |\zeta_p|^2)(1 - |\zeta_q^n|^2)}, \tag{87}
$$

$$
h_{(\zeta_p^n, \zeta_q^n)} = \frac{1 - \zeta_p^n \bar{\zeta}_q^n}{1 - \bar{\zeta}_p^n \zeta_q^n}. \tag{88}
$$

Figure 2 depicts the discrete Lobachevsky space $\mathfrak{H}_n$ for $n = 5, 112$.

4.2. The Nonassociative Discrete Geometry of S^2

Let n points be distributed randomly on the surface of the unit sphere, S^2. Consider the geodesic triangulation of the sphere, with the 2D-simplex being a geodesic triangle. We assign to each surface vertex, p, the polar coordinates (θ_p, φ_p). Without loss of generality, assume that the central vertex [i.e., the neutral
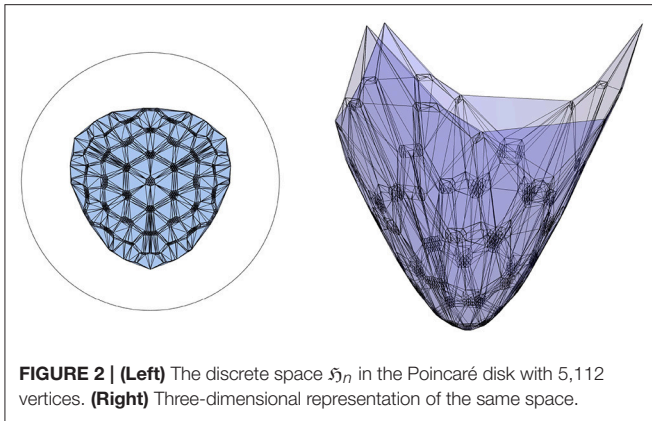
FIGURE 2 | (Left) The discrete space $\mathfrak{H}_n$ in the Poincaré disk with 5,112 vertices. **(Right)** Three-dimensional representation of the same space.

element of the loop $QS(2)$] is located at the north pole. Using stereographic projection, we obtain

$$\zeta_p = \tan\left(\frac{\theta_p}{2}\right)e^{i\varphi_p}. \tag{89}$$

The non-associative (partial) operation (86) now takes the form

$$\zeta_{pq} = \frac{\zeta_p + \zeta_q}{1 - \bar{\zeta}_p \zeta_q}. \tag{90}$$

Writing ζ_{pq} as

$$\zeta_{pq} = \tan\left(\frac{\theta_{pq}}{2}\right)e^{i\varphi_{pq}}, \tag{91}$$

one can rewrite (90) in spherical coordinates as follows:

$$\theta_{pq} = 2\tan^{-1}\left(\left|\frac{\zeta_p + \zeta_q}{1 - \bar{\zeta}_p \zeta_q}\right|\right), \tag{92}$$

$$\varphi_{pq} = \arg(\zeta_p + \zeta_q) - \frac{i}{2}\ln l(\zeta_p, \zeta_q), \tag{93}$$

where $l(\zeta_p, \zeta_q)$ stands for the associator

$$l(\zeta_p, \zeta_q) = \frac{1 - \zeta_p \bar{\zeta}_q}{1 - \bar{\zeta}_p \zeta_q}.$$

The computation of the elementary holonomy yields:

$$h_{(\zeta_p, \zeta_q)} = \frac{1 + \zeta_p \bar{\zeta}_q}{1 + \bar{\zeta}_p \zeta_q}. \tag{94}$$

We define the left-invariant metric on S^2 as

$$g(\zeta_p, \zeta_q) = \frac{4|\zeta_p - \zeta_q|^2}{(1 + |\zeta_p|^2)(1 + |\zeta_q|^2)}. \tag{95}$$

Comment 4.1. The finite loop contains hidden information concerning the geometry of the sphere. The spherical symmetry is determined by the relation between the elementary holonomy and the associator, $h_{(\zeta_p, \zeta_q)} = l_{(\zeta_p, L_{\zeta_p}^{-1}\zeta_q)}$. The smooth sphere can be regarded as a result of passage to the limit of $n \to \infty$.

4.3. Discrete de Sitter Space

Consider a finite set $M = \mathbb{Z}_n^4 = \{p = (p^\mu) \mid p^\mu \in \mathbb{Z}_n, \mu = 0, \ldots, 3, n \in \mathbb{N}\}$ where $\mathbb{Z}_n = \{p = -n, \ldots, n\}$ are integers. A partial loop $QH_{\mathbb{Z}_n^4}$ is defined as a set of quaternions

$$
\begin{aligned}
H_{\mathbb{Z}_n^4} = \quad &\{\zeta_p = \ell(p^0 + i(p^1 i + p^2 j + p^3 k)) : \ell = \text{const}, \\
&i^2 = -1, \; i \in \mathbb{C}; \; p \in \mathbb{Z}_n^4\}
\end{aligned} \tag{96}
$$

with the indefinite norm $\|\zeta_p\|^2 = \ell^2 p^\mu p_\mu$ and the binary operation $H_{\mathbb{Z}_n^4} \times H_{\mathbb{Z}_n^4} \mapsto H_{\mathbb{Z}_n^4}$ defined by

$$\zeta_p * \zeta_q = \zeta_{pq} = \left(\zeta_p + \zeta_q\right)\Big/\left(1 + \frac{K}{4}\zeta_p^+ \zeta_q\right), \quad \zeta_p, \zeta_q \in H_{\mathbb{Z}_n^4}. \tag{97}$$

We define a partial geodiodular finite space $\mathcal{M}_n$ at the neutral zero as follows:

- $QH_{\mathbb{Z}_n^4}$ is the odule with the multiplication operation induced by the quaternionic algebra over $\mathbb{Z}$,
- $\mathcal{M}^+ = H_{\mathbb{Z}_n^4}$ is the vector space induced by the quaternionic algebra over $\mathbb{Z}$.

The left-invariant metric and elementary holonomy are given by

$$g(\zeta_p, \zeta_q) = \frac{\|\zeta_p - \zeta_q\|^2}{\|1 - \frac{K}{4}\zeta_p^+ \zeta_p\|\|1 - \frac{K}{4}\zeta_q^+ \zeta_q\|}, \tag{98}$$

$$h_{(\zeta_p, \zeta_q)}\zeta_m = \left(1 - \frac{K}{4}\zeta_q \zeta_p^+\right)\zeta_m\Big/\left(1 - \frac{K}{4}\zeta_q^+ \zeta_p\right). \tag{99}$$

Taking $q = p + \delta, |\delta| \ll n$, we obtain $\zeta_q = \zeta_p + \Delta\zeta_q$, where $\Delta\zeta_q = \ell(\delta^0 + i(\delta^1 i + \delta^2 j + \delta^3 k))$. Then, the metric (98) can be recast as

$$g^{\zeta_p}(\zeta_q, \zeta_q) = \frac{\|\Delta\zeta_q\|^2}{\|1 - \frac{K}{4}\zeta_p^+ \zeta_p\|^2} + O(K\ell^2), \tag{100}$$

and

$$g^{\zeta_p}(\zeta_q, \zeta_q) \longrightarrow ds^2 = \frac{(d\zeta^0)^2 - (d\zeta^1)^2 - (d\zeta^2)^2 - (d\zeta^3)^2}{(1 - \frac{K}{4}((\zeta^0)^2 - (\zeta^1)^2 - (\zeta^2)^2 - (\zeta^3)^2)^2)^2},$$

as $\ell \to 0, \; n \to \infty$. A similar consideration of the elementary holonomy yields

$$h_{(\zeta_p, \zeta_q)}\zeta_m = \zeta_m + \frac{K}{4}\Delta(\zeta_p, \zeta_q, \zeta_m) + O\left(K\ell^2\right), \tag{101}$$

where $\Delta(\zeta_p, \zeta_q, \zeta_m) = \zeta_m \zeta_q^+ \zeta_p - \zeta_q \zeta_p^+ \zeta_m$. Approaching the limit $\ell \to 0$ and $n \to \infty$, one recovers the curvature tensor of de Sitter space:

$$R_{\mu\nu\lambda\sigma} = -\frac{K}{2}\varepsilon_{\mu\nu\kappa\delta}\varepsilon^{\kappa\delta}_{\;\;\lambda\sigma}. \tag{102}$$

Thus, smooth de Sitter space could be regarded as the passage to the limit of $\ell \to 0, n \to \infty$ and $\ell n = \text{const}$. We see that, to some extent, the information about the geometry of de Sitter spacetime is hidden in the finite loop $QH_{\mathbb{Z}_n^4}$.

4.4. Discrete Friedmann-Robertson-Walker Spacetime

Consider a finite set $M = \mathbb{Z}_n^3 = \{p = (p^i) \mid p^i \in \mathbb{Z}_n, i = 1, 2, 3, n \in \mathbb{N}\}$, where $\mathbb{Z}_n = \{p = -n, \dots, n\}$ are integers. We denote by $QH_{\mathbb{Z}_n^3}$ a partial loop defined as a set of quaternions,

$$H_{\mathbb{Z}_n^3} = \{\zeta_{\mathbf{p}} = i\ell(p^1 i + p^2 j + p^3 k) \colon \ell = \text{const},$$
$$i^2 = -1, \ i \in \mathbb{C}; \ \mathbf{p} \in \mathbb{Z}_n^3\}, \quad (103)$$

with the norm $\|\zeta_{\mathbf{p}}\|^2 = \ell^2 p^i p_i = -\ell^2\left((p^1)^2 + (p^2)^2 + (p^3)^2\right)$ and the binary operation $H_{\mathbb{Z}_n^3} \times H_{\mathbb{Z}_n^3} \mapsto H_{\mathbb{Z}_n^3}$ given by

$$\zeta_{\mathbf{p}} \star \zeta_{\mathbf{q}} = \zeta_{\mathbf{pq}} = (\zeta_{\mathbf{p}} + \zeta_q)\Big/\left(1 + \frac{K}{4}\zeta_{\mathbf{p}}^+ \zeta_q\right), \quad \zeta_{\mathbf{p}}, \zeta_{\mathbf{q}} \in H_{\mathbb{Z}_n^3}. \tag{104}$$

A partial geodiodular finite space $\mathcal{M}_n$ at the neutral element $e = 0$ is defined as

- $QH_{\mathbb{Z}_n^3}$ is the odule with multiplication induced by the quaternionic algebra over $\mathbb{Z}$,
- $\mathcal{M}_n^+ = H_{\mathbb{Z}_n^3}$ is the vector space induced by the quaternionic algebra over $\mathbb{Z}$.

Let $H_{\mathbb{Z}_n^3 \times \mathbb{T}_n}^{\mathcal{F}}$ be a set of quaternions,

$$H_{\mathbb{Z}_n^3 \times \mathbb{T}_n}^{\mathcal{F}} = \{x_{\mathbf{p}}^n = \tau_n + a(\tau_n)\zeta_{\mathbf{p}}, \ \mathbf{p} \in \mathbb{Z}_n^3, \},$$

where $\mathbb{T}_n = \{\tau_n \colon n \in \mathbb{N}\}$ and τ_n is a discrete parameter. We define the partial loop $QH_{\mathbb{Z}_n^3 \times \mathbb{T}_n}^{\mathcal{F}} \simeq QH_{\mathbb{Z}_n^3} \times \mathbb{T}_n$ as the loop with the operation $*$ given by

$$x_{\mathbf{p}}^n * x_{\mathbf{q}}^n = \tau_n + a(\tau_n)\zeta_{\mathbf{p}} \star \zeta_{\mathbf{q}}. \tag{105}$$

For the associator we have

$$l_{(\zeta_{\mathbf{p}}, \zeta_q)}\zeta_{\mathbf{m}} = \left(1 + \frac{K}{4}\zeta_q^+ \zeta_{\mathbf{p}}\right)\zeta_{\mathbf{m}}\Big/\left(1 + \frac{K}{4}\zeta_{\mathbf{p}}^+ \zeta_q\right). \tag{106}$$

We introduce the nonassociative discrete FRW spacetime as $\mathcal{M}^{\mathcal{F}} = \bigcup_{n \in \mathbb{N}} \mathcal{M}_n^{\mathcal{F}}$, where a partial geodiodular finite space $\mathcal{M}_n^{\mathcal{F}}$ consists of

- $QH_{\mathbb{Z}_n^3 \times \mathbb{T}_n}^{\mathcal{F}}$, the odule with multiplication induced by the quaternionic algebra,
- $\mathcal{M}_n^+ = H_{\mathbb{Z}_n^3} \times \mathbb{T}_n$, the vector space with structure induced by the quaternionic algebra over $\mathbb{Z}$,
- the partial binary operation for the given $x_{\mathbf{p}}^n \in \mathcal{M}_n^{\mathcal{F}}$ and $x_{\mathbf{q}}^{n'} \in \mathcal{M}_{n'}^{\mathcal{F}}$ is defined as

$$x_{\mathbf{p}}^n * x_{\mathbf{q}}^{n'} = \tau_n + \tau_{n'} + a(\tau_n + \tau_{n'})\zeta_{\mathbf{p}} \star \zeta_{\mathbf{q}}. \tag{107}$$

Thus, τ_n can be interpreted as a "discrete time" in our model .

The computation of the elementary holonomy yields

$$h_{(\zeta_{\mathbf{p}}, \zeta_{\mathbf{q}})}\zeta_{\mathbf{m}} = \left(1 - \frac{K}{4}\zeta_{\mathbf{q}}\zeta_{\mathbf{p}}^+\right)\zeta_{\mathbf{m}}\Big/\left(1 - \frac{K}{4}\zeta_{\mathbf{q}}^+ \zeta_{\mathbf{p}}\right). \tag{108}$$

One can introduce the discrete left invariant metric on $\mathcal{M}^{\mathcal{F}}$ in the same way as in Equation (98),

$$g(x_{\mathbf{p}}, x_{\mathbf{q}}') = (\tau_{n'} - \tau_n)^2 - \frac{a(\tau_n)a(\tau_{n'})\|\zeta_{\mathbf{p}} - \zeta_{\mathbf{q}}\|^2}{\|1 - \frac{K}{4}\zeta_{\mathbf{p}}^+ \zeta_{\mathbf{q}}\|\|1 - \frac{K}{4}\zeta_{\mathbf{q}}^+ \zeta_{\mathbf{p}}\|}. \tag{109}$$

Finally, we obtain the metric for FRW spacetime in the continuous limit,

$$g(\zeta_p, \zeta_q) \to ds^2 = -\frac{(d\zeta^1)^2 + (d\zeta^2)^2 + (d\zeta^3)^2}{\left(1 + \frac{K}{4}\left((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2\right)\right)^2}. \tag{110}$$

5. EMERGENCE OF SPACETIME

We are now ready to propose a discrete model of spacetime based on nonassociative geometry. In our approach we consider a triangulated topological space, where the vertices are "points" of discrete spacetime, and form the elements of the loop with a partial operation. The edges are identified as the "geodesics" connecting vertices. In contrast to the Regge model (for details and a review, see [82–85]), in nonassociative geometry the curvature does not reside at the vertices, but is defined by the elementary holonomy associated with 2-faces of simplices.

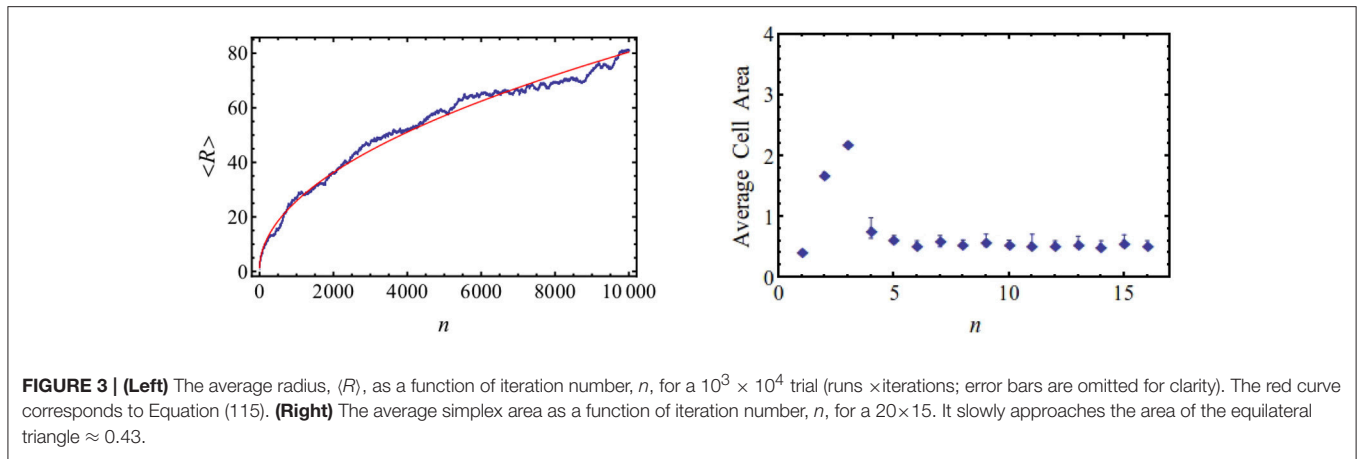Our approach is based on the following assumptions:

- Vertices of the simplices are considered as partially ordered "atoms" of discrete spacetime.
- The distance between two neighboring atoms can not be less than the fundamental length, ℓ_p.
- Interaction between atoms of spacetime, being nonlocal and nonassociative, defines the spacetime geometry.
- Spacetime geometry is encoded in the nonassociative structure of spacetime.
- The foliation of spacetime consists of space-like slices joined by time-like edges.
- Time is quantized and the evolution of spacetime geometry is governed by a random/stochastic process.
- The spacetime dimension may be dynamical.

Let $\mathfrak{M}_n$ be a triangulated topological space, with n being the number of vertices. We identify the elements $x_p \in \mathfrak{M}_n$ as the vertices of simplices and introduce a partial nonassociative operation, $x_p * x_q$. Since the introduced operation is partial, for some elements $x_p, x_q \in \mathfrak{M}_n$ the result of multiplication may not belong to $\mathfrak{M}_n$: $x_p * x_q \mapsto x_p' \notin \mathfrak{M}_n$. Adding the point x_p' to the set $\mathfrak{M}_n$ as a new vertex, we obtain $\mathfrak{M}_{n+1}$. This leads to the partial ordering

$$\cdots \prec \mathfrak{M}_n \prec \mathfrak{M}_{n+1} \prec \cdots \prec \mathfrak{M}_{n'} \dots, \tag{111}$$

and the nonassociative discrete spacetime, $\mathfrak{M} = \bigcup_{n \in \mathbb{N}} \mathfrak{M}_n$, becomes a partially ordered finite set or *causal set* ([20, 86], see section 6 for details). The evolution of spacetime geometry is interpreted in terms of the sequence: $\cdots \to \mathfrak{M}_n \to \mathfrak{M}_{n+1} \to \cdots \to \mathfrak{M}_{n'} \dots$.

Comment 5.1. Albeit the fundamental length, ℓ_p, is not necessarily equal to the Planck length, we will take it to be so

FIGURE 3 | (Left) The average radius, $\langle R \rangle$, as a function of iteration number, n, for a $10^3 \times 10^4$ trial (runs $\times$ iterations; error bars are omitted for clarity). The red curve corresponds to Equation (115). **(Right)** The average simplex area as a function of iteration number, n, for a 20×15. It slowly approaches the area of the equilateral triangle ≈ 0.43.

from this point forward. The known problem of "many-fingered time" [4–8] is overcome in our model as follows. In contrast to CDT, in our approach edges are allowed to be longer than the Planck length. This implies that the local "time" between two slices depends on the 'coordinates' (vertices), and each realization of the random process yields different results. Thus, history becomes foliation dependent (i.e., the many-fingered time problem). However, actual physical quantities are defined by probabilities, which are foliation independent. Since in our approach the evolution of the spacetime geometry is governed by a stochastic dynamical process, we avoid the foliation dependence problem.

To illustrate our approach, below we consider a cosmological toy model of (2+1)-D discrete spacetime and a discrete FRW cosmological model.

5.1. Toy Model of (2+1)-D Discrete Spacetime

In this section we consider a simple model based on the non-associative compact space $\mathfrak{S}(\tau_n) \simeq Q\mathfrak{S}(\tau_n)$, obtained by triangulation of the partial loop, $Q\mathfrak{S}(\tau_n)$, defined as follows. An arbitrary element, $\zeta_p^n \in Q\mathfrak{S}(\tau_n)$, takes the form $\zeta_p^n = R_n \zeta_p$, where $R_n = R(\tau_n)$ is the "radius" of $\mathfrak{S}(\tau_n)$, τ_n is the discrete intrinsic time, a quantum of time is $\Delta \tau_n = \tau_{n+1} - \tau_n = 1$ in Planck units, and $\zeta_p \in \mathbb{C}$ is the p-th complex number.

The partial nonassociative operation preserving the geometry of spacetime is given by

$$\zeta_{pq}^n = \frac{\zeta_p^n + \zeta_q^n}{1 - \bar{\zeta}_p^n \zeta_q^n / R_n^2}, \quad \zeta_p^n, \zeta_q^n, \zeta_{pq}^n \in Q\mathfrak{S}(\tau_n), \quad (112)$$

$$\zeta_{pq}^{n+1} = \frac{R_{n+1}}{R_n} \zeta_{pq}^n, \quad \zeta_{pq}^{n+1} \in Q\mathfrak{S}(\tau_{n+1}). \quad (113)$$

We assume that evolution of the discrete spacetime geometry is a stochastic process. The radius R_n is defined by a random walk with step ℓ_p and reflecting barrier R_0 at $n = 0$ [87–89]. After n

steps and for $n \gg 1$, the expectation value of the scale factor will grow as

$$\langle R \rangle = R_0 + \ell_p \sqrt{\frac{2}{\pi n}} \int_0^\infty x e^{-x^2/2n}. \quad (114)$$

The computation yields

$$\langle R \rangle = R_0 + \sqrt{2n/\pi}\, \ell_p. \quad (115)$$

In **Figure 3** we present the results of our numerical simulations. As one can see (left panel) the analytical formula (115) is in good agreement with our numerical results. We find that, for large n, the average area of each 2D-simplex converges to the area of the equilateral triangle, $A = \sqrt{3}/4 \approx 0.433$. (see **Figure 3**, right panel.)

Assume that $\mathfrak{S}(\tau_n)$ has m points. The map $Q\mathfrak{S}(\tau_n) \mapsto Q\mathfrak{S}(\tau_{n+1})$ defined by Equation (113) yields m^2 points belonging to $Q\mathfrak{S}(\tau_{n+1})$. To obtain $\mathfrak{S}(\tau_{n+1})$ one should triangulate the set $Q\mathfrak{S}(\tau_{n+1})$. During the triangulation only the points with distance $\ell \geq \ell_p$ between closest neighbors will survive. After triangulation, we are left with the two space-like slices $\mathfrak{S}(\tau_n)$ and $\mathfrak{S}(\tau_{n+1})$. Next, we triangulate the set $\mathfrak{S}(\tau_n) \bigcup \mathfrak{S}(\tau_{n+1})$, joining the vertices of the spatial slices by time-like edges. Repeating this procedure, we obtain a partially ordered discrete spacetime, $\mathfrak{S} = \bigcup_n \mathfrak{S}(\tau_n): \mathfrak{S}(\tau_0) \prec \cdots \prec \mathfrak{S}(\tau_n) \prec \mathfrak{S}(\tau_{n+1}) \prec \ldots$.

In **Figures 4–7** we present the results of numerical simulations showing a particular realization of the random process. As an initial state at the moment $\tau_0 = 0$, we take a discrete space $\mathfrak{S}(0)$ defined by the tetrahedron formed by 2-dimensional equilateral simplices of the same size, with distance ℓ_p between neighboring vertices and $R_0 = \sqrt{3/8}\ell_p$. As the number of iterations increases, the triangulation of $\mathfrak{S}(\tau_n)$ becomes finer and its elements approach equilateral 2D-simplices (see **Figure 4**). As one can see, with growing n the simplicial complex is approximated by S^2. The results of our numerical simulations illustrating the evolution of spacetime geometry are presented in **Figure 5**. In **Figure 6** the triangulation of the spacetime, $\mathfrak{S} = \bigcup_n \mathfrak{S}(\tau_n)$ for $n = 5$

is presented. The spatial foliation of the spacetime geometry is depicted in **Figure 7**. The correspondence between simplicial complexes in the complex plane, $\mathbb{C}$, and $\mathfrak{S}(\tau_n)$ is shown on the right side of the figure. The surfaces $\tau_n = \text{const}$ and $\tau_{n+1} = \text{const}$ represent adjacent two-dimensional spaces. The timelike edges are omitted for clarity.

5.2. Discrete Spacetime: FRW Cosmological Model

Here we consider a cosmological discrete spacetime model based on the discrete nonassociative FRW geometry described in section 4.4. We start with the partial loop $\mathrm{QH}_{\mathbb{Z}_n^3}$ based on the set of quaternions

$$\mathrm{H}_{\mathbb{Z}_n^3} = \{\zeta_{\mathbf{p}} = \mathrm{i}\ell_p(p^1 i + p^2 j + p^3 k):$$
$$\mathrm{i}^2 = -1,\ \mathrm{i} \in \mathbb{C};\ \ \mathbf{p} = (p^1, p^2, p^3),\ \mathbf{p} \in \mathbb{Z}_n^3\}. \quad (116)$$

Let $\zeta_{\mathbf{p}}^n = R(\tau_n)\zeta_{\mathbf{p}}$ be an element of the loop $\mathrm{QH}_{\mathbb{Z}_n^3}$, where τ_n is a discrete intrinsic time, $R(\tau_n) = a(\tau_n)\ell_p$ with $a(\tau_n)$ being the scale
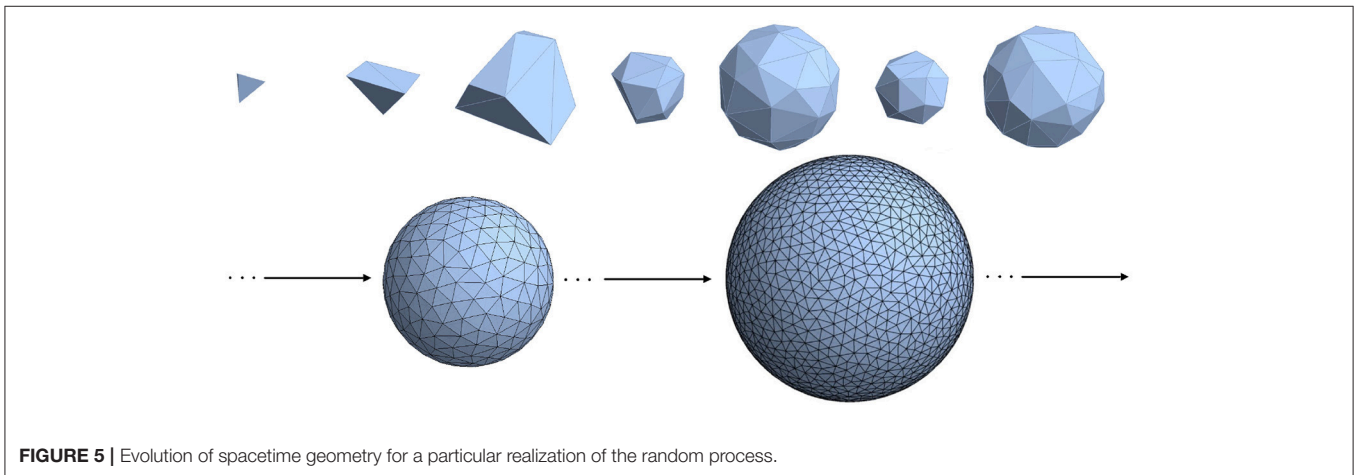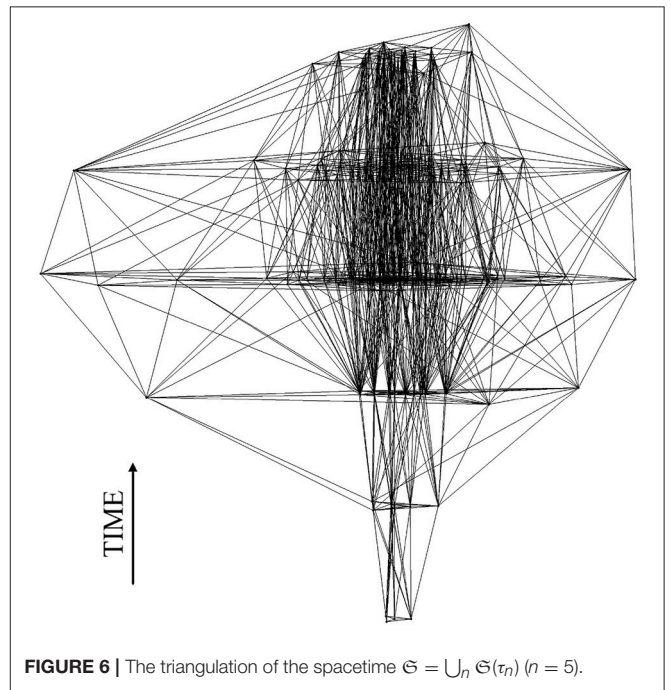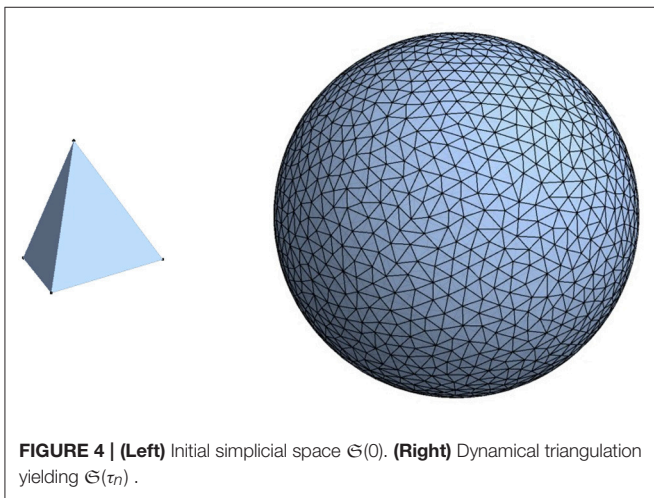
factor. We define the partial binary operation in $\mathrm{QH}_{\mathbb{Z}_n^3}$ as follows:

$$\zeta_{\mathbf{pq}}^n = R(\tau_n)\big(\zeta_{\mathbf{p}} + \zeta_{\mathbf{q}}\big)\Big/\Big(1 + \frac{K}{4}\,\zeta_{\mathbf{p}}^+\,\zeta_{\mathbf{q}}\Big), \quad \zeta_{\mathbf{p}},\,\zeta_{\mathbf{q}} \in \mathrm{H}_{\mathbb{Z}_n^3}, \tag{117}$$

$$\zeta_{\mathbf{pq}}^{n+1} = \frac{R(\tau_{n+1})}{R(\tau_n)}\,\zeta_{\mathbf{pq}}^n, \quad \zeta_{\mathbf{pq}}^n \in \mathrm{QH}_{\mathbb{Z}_n^3},\ \zeta_{\mathbf{pq}}^{n+1} \in \mathrm{QH}_{\mathbb{Z}_{n+1}^3}, \tag{118}$$

where "/" denotes the right division of the quaternions.

Let $\mathfrak{M}(\tau_n)$ be a triangulated topological space obtained by triangulation of $\mathrm{QH}_{\mathbb{Z}_n^3}$. Assume that $\mathfrak{M}(\tau_n)$ has m points. Then the map $\mathfrak{M}(\tau_n) \mapsto \mathfrak{M}(\tau_{n+1})$ defined by Equation (118) yields m^2 points belonging to $\mathrm{QH}_{\mathbb{Z}_{n+1}^3}$. However, only the points with distance $\ell \geq \ell_p$ between closest neighbors survive during

FIGURE 4 | **(Left)** Initial simplicial space $\mathfrak{S}(0)$. **(Right)** Dynamical triangulation yielding $\mathfrak{S}(\tau_n)$.

FIGURE 6 | The triangulation of the spacetime $\mathfrak{S} = \bigcup_n \mathfrak{S}(\tau_n)$ $(n = 5)$.

FIGURE 5 | Evolution of spacetime geometry for a particular realization of the random process.
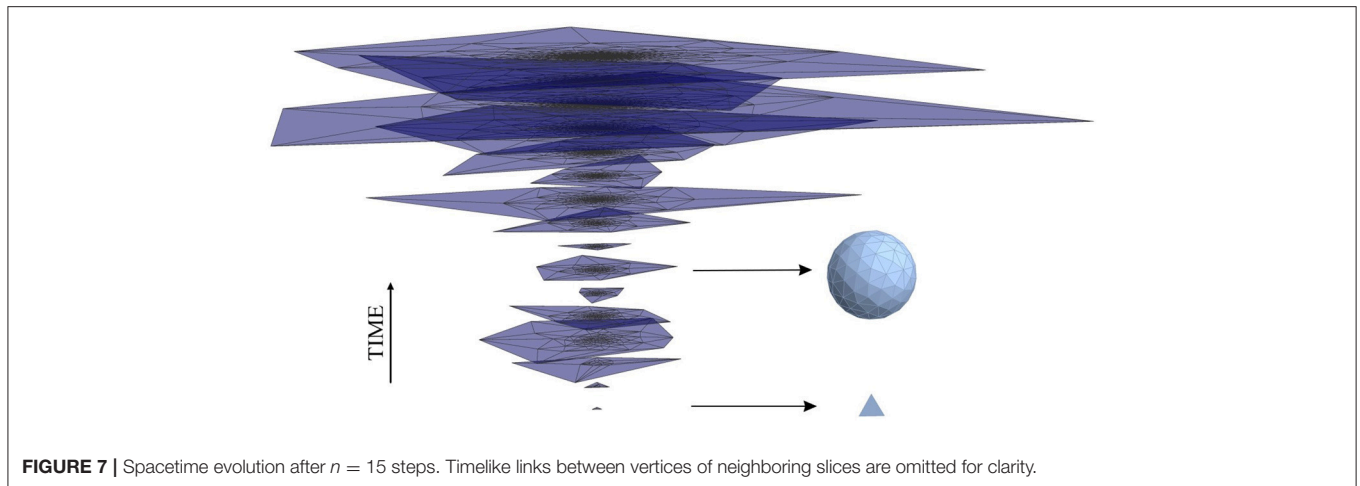
FIGURE 7 | Spacetime evolution after $n = 15$ steps. Timelike links between vertices of neighboring slices are omitted for clarity.
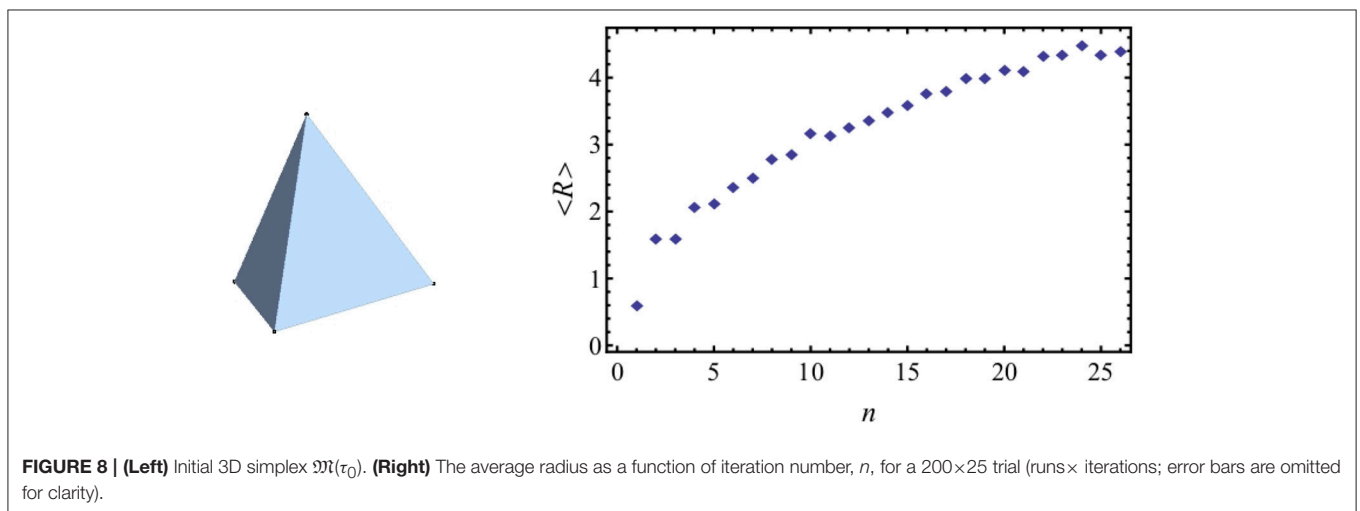

FIGURE 8 | (Left) Initial 3D simplex $\mathfrak{M}(\tau_0)$. **(Right)** The average radius as a function of iteration number, n, for a 200×25 trial (runs× iterations; error bars are omitted for clarity).

the triangulation of $QH_{\mathbb{Z}_{n+1}^3}$. After triangulation, we obtain two space-like slices, $\mathfrak{M}(\tau_n)$ and $\mathfrak{M}(\tau_{n+1})$. The next step is to triangulate the set $\mathfrak{M}(\tau_n) \bigcup \mathfrak{M}(\tau_{n+1})$, joining the vertices of spatial slices by time-like edges. Repeating this procedure, we obtain a partially ordered discrete spacetime: $\mathfrak{M}(\tau_0) \prec \cdots \prec \mathfrak{M}(\tau_n) \prec \mathfrak{M}(\tau_{n+1}) \prec \ldots$. Since some information is lost during triangulation in the form of "missing" vertices (those that would have been closer than ℓ_p), the evolution is irreversible. Thus, in our approach the arrow of time, being related to the partial ordering, appears naturally [18].

We assume that evolution of the discrete spacetime geometry is a stochastic process. The scale factor, R_n, is defined by a random walk with a step ℓ_p and reflecting barrier R_0 at $n = 0$. After n steps, the average scale factor will grow as $\langle R_n \rangle = R_0 + \sqrt{2n/\pi}\, \ell_p$ (see **Figure 8**, right panel).

After $\sim 10^8$ steps in Planck units of time, which corresponds to pre-inflationary time, $t_i \sim 10^{-35}$ sec (time of the beginning of inflation), the size of the universe in our model would be $R_{pr} \sim 10^{-29}$cm. The size of the pre-inflationary universe in the standard FRW-model can be estimated using the relationship

between the redshift and the cosmological scale factor, $z_i = a(t)/a(t_i) - 1$, where t denotes the actual age of the universe. To estimate the redshift, we use the ΛCDM model. For the dark energy dominated era the redshift is given by Ryden [90] and Frieman et al. [91]

$$1 + z = \frac{a(t)}{a(t_m)} = \frac{\sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}\, t\right)}{\sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}\, t_m\right)}, \tag{119}$$

where $\Omega_\Lambda = \Lambda/(3H_0^2)$. This expression is quite accurate for $t > t_m = 10\,\text{Myr}(10^7\,\text{y}) = 3.15 \cdot 10^{14}$ sec. For a radiation-dominated era, we have $a(t) \propto \sqrt{t}$, and for a matter-dominated universe, $a(t) \propto t^{2/3}$. During inflation the universe is blown up to enormous size, and the changes in the cosmological scale factor are of order $a(t_i)/a(t_f) \sim e^{60} \approx 10^{26}$, where t_f denotes the time of the end of inflation. Combining all results, we obtain

$$z_i \approx 10^{26} \sqrt{\frac{t_r}{t_f}} \left(\frac{t_m}{t_r}\right)^{2/3} \frac{\sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}\, t\right)}{\sinh^{2/3}\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda}\, t_m\right)}. \tag{120}$$

Inserting $t_f \approx 10^{-33}$s, $t_r \approx 10^{14}$s, $t \approx 4.35 \cdot 10^{17}$s (age of the universe), $\Omega_\Lambda = 0.7$ and $H_0 = 2.33 \cdot 10^{-18}$s^{-1} into this expression, we obtain $z_i \simeq 10^{52}$. The pre-inflationary size of the universe can be estimated as $R_i = R(t)/z_i$.

If we choose $R(t)$ as the observable radius of the universe, $R(t) \sim 10^{28}$ cm, then the radius of the pre-inflationary universe is $R_i \sim 10^{-24}$ cm. Comparing this result with the prediction of our model, we find $R_i/R_{pr} = 10^5$. Thus, starting with only one initial mini-universe, we cannot reach an appropriate size of the universe at the beginning of the inflationary epoch. To overcome this difficulty, one can assume that spacetime would consist of many small fluctuating simplicial regions.

These fluctuations may form mini-clusters, as illustrated in **Figure 9**, created in different regions and may grow as "mini-universes" (**Figure 10**). We consider all these mini-universes as a multiverse [92, 93]. After $\sim 10^7 - 10^8$ steps from the very beginning, the mini-universes grow enough and may merge in different regions of space to form pre-inflationary universes (bubbles) with appropriate average size $\sim 10^{28}$ cm. This multiverse state can be metastable and, as a result of the first-order phase transition, the nucleation of bubbles may occur and lead to inflation. Indeed, if the duration of the phase transition is short enough, bubble nucleation of the new phase may have an exponential character [94–96]. Thus, in our model the post-inflationary Universe is the result of first-order inflation, occurring during a strong first-order phase transition [97–99] (**Figure 11**).
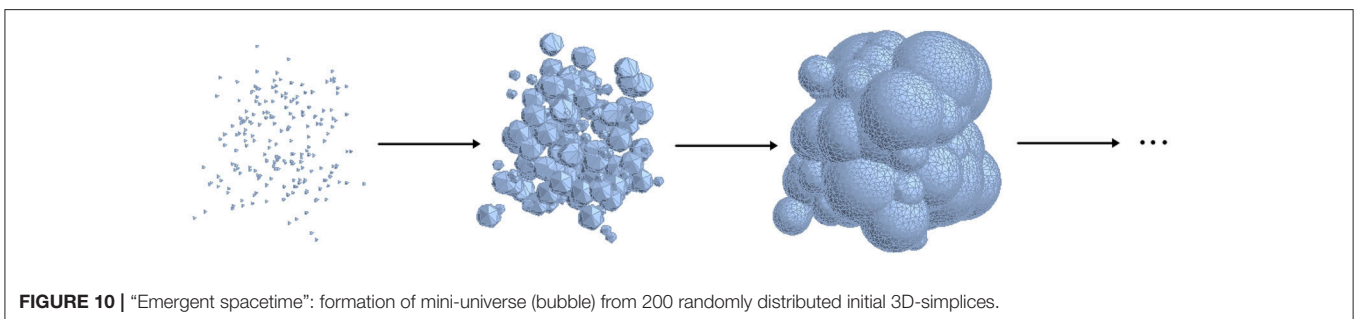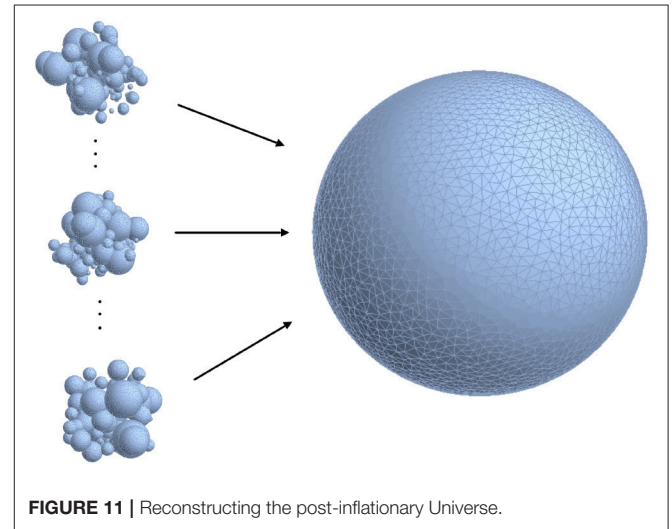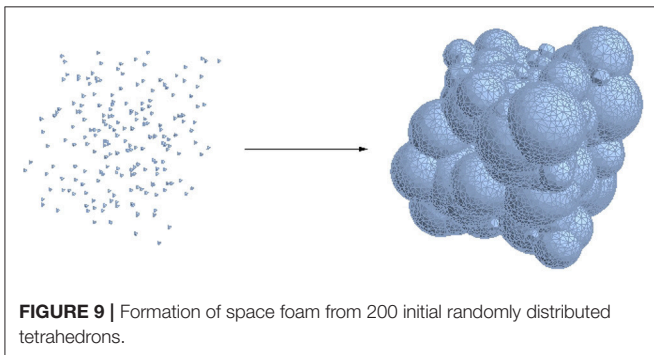
Comment 5.2. An alternative scenario of arising spacetime structure may be formation of 3D-clusters from 2D-simplicial complexes as illustrated in **Figure 9**. This implies that the 2D-simplices are more fundamental than 3D-simplices, and 3D volume effectively is created from random foam-like structures. In this scenario the spacetime dimension is dynamical and may vary depending on fusion of simplicial complexes.

6. DISCUSSION AND CONCLUSIONS

The preceding examples show how the framework of nonassociative geometry may describe the discrete structure of spacetime. We deal with points ("atoms") of discrete nonassociative spacetime. In our approach, at the Planckian scale the standard concept of spacetime is replaced by the nonassociative discrete structure. At large spacetime scales this "looks like" a smooth manifold, and the information about the geometry of the continuous spacetime is hidden in the structure of the discrete nonassociative spacetime.

In our approach it is assumed that the "universe" is created from a few initial points. This leads to a new understanding of the "big-bang": the initial singularity does not exist. Spacetime evolution is described by a stochastic process preserving spacetime geometry. As a result, we obtain a discrete spacetime with many of the attributes of continuum spacetime: spacelike slices, "many-fingered time," and causal structure. Note that, while the history of spacetime is foliation dependent, the

FIGURE 11 | Reconstructing the post-inflationary Universe.

FIGURE 9 | Formation of space foam from 200 initial randomly distributed tetrahedrons.

FIGURE 10 | "Emergent spacetime": formation of mini-universe (bubble) from 200 randomly distributed initial 3D-simplices.

observables are foliation independent since they are defined by probabilities.

If we admit that the fluctuations can occur in different regions, then foam-like structures can be created and form a spatial foam (**Figure 9**). This implies that the dimension of spacetime is dynamical and 3D-clusters are created from 2D-simplicial complexes like a volume can be created from foam-like structures. This scenario is very speculative, but deserves further study.

Existence of the fundamental length as a minimal admissible length scale in the Universe implies that the emergence of space in the expanding cosmological models is a continuous process. This point of view is supported by the models with eternal inflation. In these models the inflationary phase of the Universe's expansion lasts forever in at least some regions of the Universe [93, 97].

We would like to compare our approach with relevant models of discrete spacetime proposed in CS [19–21] and CDT [14, 22–28]. Both theories attempt, as their names suggest, to preserve the causality present in continuous spacetime while also constructing a discrete underlying structure. We briefly recount the main features of these approaches to compare and contrast them with our model.

CS proposes that the microstructure of spacetime is a partially ordered set (causal set or causet). A *causal set* is a partially ordered set of elements $\mathcal{C}$ (sometimes also abbreviated a *poset*) together with the relation $\prec$, called "precedes," such that 1) if $x \prec y$ and $y \prec z$, then $x \prec z$, $\forall x, y, z \in \mathcal{C}$; 2) if $x \prec y$ and $y \prec x$, then $x = y$, $\forall x, y, z \in \mathcal{C}$; 3) for any pair of elements x and z in $\mathcal{C}$, the set of other elements $\{y \mid x \prec y \prec z\}$ between x and z is finite. Thus, the partial order encodes information about the causal structure of spacetime.

These axioms establish the transitivity, non-circularity, and finiteness of causal sets. Thus, the elements of $\mathcal{C}$ lie as nodes along branches of links between them, with each element representing an abstraction of a fundamental unit of spacetime. The fundamental hypothesis of CS theory is that spacetime is described by a causet made of elements of Planckian size in the quantum regime. The order of the set gives rise to the causality present in macroscopic spacetime, whereas the number of elements determines the spacetime's volume. Together, these two features combine to produce the macroscopic geometry of spacetime.

Like CS, CDT is also concerned with preserving causality at a fundamental scale but is more explicitly geometrical. Formally, CDT seeks a discrete version of Feynman path integral [22, 28],

$$Z = \int \mathcal{D}g e^{iS[g]} \rightarrow Z_a, \qquad (121)$$

where the integration is performed over all Lorentzian geometries with topology $\Sigma \times [0, 1]$, with Σ being a compact, connected three-dimensional manifold. The fundamental building blocks of CDT are equilateral 4-simplices of side length a, that is, four-dimensional generalizations of equilateral tetrahedra. These simplices are arranged such that their spacelike edges form spacelike surfaces separated by their timelike edges. More

explicitly, the partition function, Z_a is given by

$$Z_a = \frac{1}{C_T} \sum_T e^{iS[T]}, \qquad (122)$$

where $S[T]$ is a suitable gravitational action, and the sum is over the chosen set of abstract triangulations of $\mathcal{M}$.

So far, CDT has relied on Monte Carlo techniques to build fluctuating spacetimes out of simplices. It should also be stressed that the minimum length cutoff at a is *not* identified with the Planck length, but is used as part of a limiting procedure to obtain a continuum spacetime. We refer the interested reader to the cited bibliography and references therein for promising results and developments in both CS and CDT, which would be out of scope for our present discussion.

The description of discrete spacetime by nonassociative geometry is both similar and distinct from those by CS and CDT. Like these two approaches, nonassociative geometry incorporates causality as a fundamental axiom of the model. A given simplicial spacetime is in a precedence relation to the one that follows it, $\mathfrak{M}(\tau_n) \prec \mathfrak{M}(\tau_{n+1})$. A temporal direction is established and can be represented by timelike edges between nodes belonging to times τ_n and τ_{n+1}. Since vertices in $\mathfrak{M}(\tau_n)$ determine those in $\mathfrak{M}(\tau_{n+1})$ but, due to the minimum length restriction, not vice versa, a definite arrow of time is established. However, unlike the stochastic sequential growth used in CS, where different branches of spacetime grow one element at a time in each discrete time interval, nonassociative geometry elements are created simultaneously in a non-local manner. And unlike CDT, timelike links in nonassociative geometry can have variable length. This yields the partial ordering of spacetimes,

$$\cdots \prec \mathfrak{M}(\tau_n) \prec \mathfrak{M}(\tau_{n+1}) \prec \cdots \prec \mathfrak{M}(\tau_{n'}) \ldots, \qquad (123)$$

and the nonassociative discrete spacetime, $\mathfrak{M} = \bigcup_{n \in \mathbb{N}} \mathfrak{M}(\tau_n)$, becomes a finite poset as described in section 5. The evolution of spacetime geometry is interpreted in terms of a sequence: $\cdots \rightarrow \mathfrak{M}(\tau_n) \rightarrow \mathfrak{M}(\tau_{n+1}) \rightarrow \cdots \rightarrow \mathfrak{M}(\tau_{n'}) \ldots$. Thus, in our approach the arrow of time appears naturally. The other difference with CDT is that, in contrast to CDT, in our model the curvature, being defined by the elementary holonomy associated with the faces of 2D-simplices, is nonlocal.

Simplicial cells are fundamental components of nonassociative discrete spacetimes, as in CDT. However, our numerical implementation of nonassociative geometry explicitly adopts the Planck length, ℓ_p, as a minimum cutoff length, unlike CDT. Also unlike CDT, nonassociative geometry does not restrict simplices to be equilateral, although the realization we present does converge to this scenario in the limit of high iteration numbers (i.e., large values of time). Another difference with CDT is that spacetime evolution in nonassociative geometry, while having a certain stochastic character, does not rely on Monte Carlo techniques to generate fluctuations in spacetimes, since the "random" component in the evolution resides completely in the random-walk behavior of R.

In conclusion, we have shown how one can describe continuum and discrete spacetimes in the framework of the nonassociative geometry. The first feature of our model of

discrete spacetime is the obvious absence of an initial singularity. The other important feature is a natural emergence of causal structure and an arrow of time. The irreversibility of evolution is caused by a random process and information loss due to existence of the minimal fundamental Planck length. Certainly, the loopuscular structures used in section 5 are given in an *ad hoc* manner. The open question is then how to describe more general models, using the sum over all possible geometries as in the CDT approach. This is the subject of the future study.

AUTHOR CONTRIBUTIONS

AN and HM designed the research and wrote the text. HM wrote the code and prepared figures. All authors reviewed the manuscript.

SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2019.00032/full#supplementary-material

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