



Analytical Solution of Generalized Space-Time Fractional Advection-Dispersion Equation via Coupling of Sumudu and Fourier Transforms

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The objective of this article is to present the computable solution of space-time advection-dispersion equation of fractional order associated with Hilfer-Prabhakar fractional derivative operator as well as fractional Laplace operator. The method followed in deriving the solution is that of joint Sumudu and Fourier transforms. The solution is derived in compact and graceful forms in terms of the generalized Mittag-Leffler function, which is suitable for numerical computation. Some illustration and special cases of main theorem are also discussed.

Keywords: space-time fractional advection-dispersion equation, Fourier transforms, Sumudu transforms, Hilfer-Prabhakar fractional derivative, fractional laplacian operator, Mittag-Leffler function

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INTRODUCTION

In the last decade, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics, biology, engineering, and other areas. Several numerical and analytical methods have been developed to study the solutions of nonlinear fractional partial differential equations, for details, refer to the work in [1–6]. Fractional equations have enabled the investigation of the nonlocal response of multiple phenomena such as diffusion processes, electrodynamics, fluid flow, elasticity, and many more. Nowadays, fractional derivatives have gained a significant development to model some real life phenomena in the form of partial differential equations or the ordinary equations. Several researchers have performed the numerical simulation for fractional problems and revealed their applications in different directions include [7–12] and references therein. The exchange of heat, mass and momentum are considered to be the fundamental transfer phenomena in the universe. The mathematical framework for heat and mass transfer are of same kind, basically encompass by advection-dispersion equation. In recent work many authors have demonstrated the depth of mathematics and related physical issues of advection-dispersion equations. Schumer et al. [13] gave physical interpretation of space-time fractional advection-dispersion equation. Space-time fractional advection-dispersion equations are generalizations of classical advection-dispersion equations. The use of Hilfer-Prabhakar fractional derivative operator is gaining importance in physics because of their specific properties. The objective of this paper is to derive the solution of Cauchy type generalized fractional advection

dispersion equation (18), associated with the Hilfer-Prabhakar fractional derivative. This paper provides an elegant extension of results, given earlier by Haung and Liu [14], Haubold et al. [15], Saxena et al. [16], and Agarwal et al. [17].

RESULTS REQUIRED IN THE SEQUEL

In early 90s, Watugala [18] introduced Sumudu transform, which is defined as,

$$A = \left\{ f(t) / \exists M, \tau_i > 0, i = 1, 2 \mid |f(t)| \leq M e^{\frac{|t|}{\tau_j}} \text{ if } t \in (-1)^j \times [0, \infty) \right\}. \tag{1}$$

for all real $t \geq 0$ the Sumudu transform of function $f(t) \in A$ is defined as,

$$S[f(t); u] = F(u) = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t) dt, \quad u \in (-\tau_1, \tau_2) \tag{2}$$

inversion formula of (2), is given by

$$S^{-1}[F(u)] = f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{t}{u}} F(u) du, \tag{3}$$

where γ being a fixed real number.

Among others, the Sumudu transform was shown to have units preserving properties, and hence may be used to solve problems without resorting to the frequency domain. Further details and properties about this transform can be found in Belgacem [19], Belgacem et al. [20], and Katatbeh and Belgacem [21].

For a function $u(x, t)$, the Fourier transform of with respect to x is defined by

$$F[u(x, t)] = u^*(\eta, t) = \int_{-\infty}^\infty e^{i\eta x} u(x, t) dx, \quad (-\infty < \eta < \infty) \tag{4}$$

and for the function $u^*(\eta, t)$, inverse Fourier transform with respect to η is given by the formula

$$F^{-1}[u^*(\eta, t)] = u(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\eta x} u^*(\eta, t) d\eta. \tag{5}$$

For more details of Fourier transform, see [Debnath and Bhatta [22]].

Mittag-Leffler function of two parameters is studied by Wiman [23] as

$$E_{\alpha, \beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in C, R(\alpha) > 0. \tag{6}$$

Mittag-Leffler function of three parameter introduced by Prabhakar [24] as

$$E_{\alpha, \beta}^\gamma(z) = \sum_{n=0}^\infty \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad \alpha, \beta, \gamma \in C, R(\alpha) > 0. \tag{7}$$

Riemann-Liouville fractional integral (right-sided) of order α is defined in [25]

$$I_a^\alpha(u(x, t)) = {}^RL D_t^{-\alpha}(u(x, t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} u(x, \tau) d\tau, \tag{8}$$

$(t > a), R(\alpha) > 0.$

The right sided Riemann-Liouville fractional derivative of order α defined as

$${}^RL D_t^\alpha(u(x, t)) = \left(\frac{d}{dt}\right)^n (I_a^{n-\alpha} u(x, t)) \quad (R(\alpha) > 0, n = [R(\alpha)] + 1), \tag{9}$$

here $[x]$ is the integral part of x .

Caputo [26], introduced fractional derivative of order $R(\alpha) > 0$ as

$${}_0^C D_t^\alpha(u(x, t)) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u^m(x, \tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \\ m-1 < \alpha \leq m, R(\alpha) > 0, m \in N, \\ \frac{\partial^m}{\partial t^m} u(x, t), \quad \text{if } \alpha = m, \end{cases} \tag{10}$$

The Sumudu transform of (10) is given in [27], as

$$S[{}_0^C D_t^\alpha u(x, t); s] = s^{-\alpha} \bar{u}(x, s) - \sum_{k=0}^{m-1} \frac{u^{(k)}(x, 0)}{u^{\alpha-k}}, \tag{11}$$

$(m-1 < \alpha \leq m)$

where $\bar{u}(x, s)$ is the Sumudu transform of $u(x, t)$.

Hilfer [28], gave a fractional derivative operator of two parameters μ and ν , which is generalization of (9) and (10), in the form

$${}_0 D_{0+}^{\mu, \nu}(u(x, t)) = I_t^{\nu(1-\mu)} \frac{\partial}{\partial t} \left(I_{0+}^{(1-\nu)(1-\mu)} u(x, t) \right), \quad 0 < \mu < 1 \text{ and } 0 \leq \nu \leq 1 \tag{12}$$

For $\nu = 0$, equation (12) reduces into (9) and for $\nu = 1$, equation (12) reduces into (10).

The Sumudu transform of (12) is given in [29], as

$$S[{}_0 D_{0+}^{\mu, \nu}(u(x, t)); s] = s^{-\alpha} \bar{u}(x, s) - \sum_{k=0}^{m-1} s^{k-m+\nu(m-\mu)} \frac{\partial^k}{\partial x^k} \left(I_{0+}^{(1-\nu)(1-\mu)} u(x, 0+) \right), \tag{13}$$

$(m-1 < \mu \leq m).$

Where the initial value term $I_{0+}^{(1-\nu)(1-\mu)} u(x, 0+)$ involves the Riemann-Liouville fractional derivative operator of order $(1-\nu)(1-\mu)$ as $t \rightarrow 0+$.

A generalization of Hilfer derivate is given in [30], known as Hilfer-Prabhakar derivative, is defined as:

Let $\mu \in (0, 1)$, $\nu \in [0, 1]$, and let f belongs to the set of locally integrable real valued functions i.e., $f \in L^1[0, b]$, $0 < t < b \leq \infty, f * e_{\rho, (1-\nu), \omega}^{-\gamma(1-\nu)}(\cdot) \in AC^1[0, b]$. The Hilfer-Prabhakar derivative is defined by

$${}_0 D_{\rho, \omega, 0+}^{\gamma, \mu, \nu}(u(x, t)) = E_{\rho, \nu(1-\mu), \omega, 0+}^{-\gamma \nu} \left(E_{\rho, (1-\nu)(1-\mu), \omega, 0+}^{-\gamma(1-\nu)} \right)$$

$$u(x, 0+), \tag{14}$$

where $\gamma, \omega \in R, \rho > 0$, and where $E_{\rho,0,\omega,0+}^0 f = f$. We observe that (14) reduces to the Hilfer derivative for $\gamma = 0$. The Sumudu transform of this derivative operator (14) is given in [31], in the form:

$$S \left[{}_0D_{\rho,\omega,0+}^{\gamma,\mu,\nu} (u(x, t)); s \right] = s^{-\mu} (1 - \omega s^\rho)^\gamma \bar{u}(x, s) - s^{\nu(1-\mu)-1} (1 - \omega s^\rho)^{\gamma\nu} \left[E_{\rho,(1-\nu)(1-\mu),\omega,0+}^{-\gamma(1-\nu)} (x, 0+) \right] \tag{15}$$

For details of this derivative, refer to the work in [30, 31].

Brockmann and Sokolov [32], defined a fractional Laplace operator as:

$$\Delta^{\frac{\lambda}{2}} = \frac{1}{2 \cos\left(\frac{\pi\lambda}{2}\right)} \left\{ -\infty D_x^\lambda + x D_\infty^\lambda \right\}, \quad (0 < \lambda \leq 2),$$

where the operators are defined by

$$-\infty D_x^\lambda (u(x)) = \frac{1}{k - \lambda} \int_{-\infty}^x \frac{u^k(u)}{(x - u)^{\lambda+1-k}} du, \quad (k = [\lambda] + 1),$$

and

$$x D_\infty^\lambda (u(x)) = \frac{1}{k - \lambda} \int_x^\infty \frac{u^k(u)}{(x - u)^{\lambda+1-k}} du, \quad (k = [\lambda] + 1).$$

The Fourier transform of $\Delta^{\frac{\lambda}{2}}$ is given in [32], as

$$F \left\{ \Delta^{\frac{\lambda}{2}} (u(x, t)); k \right\} = -|k|^\lambda F \{ u(x, t) \}, \quad (0 < \lambda \leq 2). \tag{16}$$

Inverse Sumudu transform of the following function is directly applicable in this sequel:

In the complex plane C , for any $R(\alpha) > 0, R(\beta) > 0$, and $\omega \in C$

$$S^{-1} \left[u^{\gamma-1} (1 - \omega u^\beta)^{-\delta} \right] = t^{\gamma-1} E_{\beta,\gamma}^\delta (\omega t^\beta). \tag{17}$$

SPACE-TIME FRACTIONAL ADEVECTION-DISPERSION EQUATION

Here we will find, the solution of the generalized space-time Advection-Dispersion equation (18) under the conditions given in (19) and (20). Our main findings in the form of the following Theorem 3.1 and Corollary 3.2.

Theorem 3.1. Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$${}_0D_{\rho,\omega,t}^{\gamma,\mu,\nu} (u(x, t)) = -\eta D_x u(x, t) + \zeta \Delta^{\frac{\lambda}{2}} (u(x, t)), \tag{18}$$

where $\lambda \in (0, 2], x \in R, t \in R^+, \mu \in (0, 1), \nu \in [0, 1]$, with initial condition,

$$E_{\rho,(1-\nu)(1-\mu),\omega,0+}^{-\gamma(1-\nu)} u(x, 0+) = g(x), \quad \gamma, \omega, x \in R, \rho > 0, \tag{19}$$

and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0, \tag{20}$$

where $\Delta^{\frac{\lambda}{2}}$ is the Laplace operator of fractional order $\lambda, \lambda \in (0, 2]$. The positive constant η represent the average fluid velocity and ζ (positive constant) represent the dispersion coefficient. Subject to the above constraints, solution of equation (18), is

$$u(x, t) = \sum_{n=0}^\infty \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^\infty e^{-ikx} g(k) \left(i\eta k - \zeta |k|^\lambda \right)^n E_{\rho,\nu(1-\mu)+n\mu}^{\gamma(n-\nu)} (\omega t^\rho) dk. \tag{21}$$

Proof: First, take the Fourier transform of equation (18) with respect to the space variable x , then

$${}_0D_{\rho,\omega,t}^{\gamma,\mu,\nu} (u^*(k, t)) = \eta i k u^*(k, t) - \zeta |k|^\lambda u^*(k, t), \tag{22}$$

$u^*(k, t)$ represent Fourier transform of $u(x, t)$. Again, apply Sumudu transform on (22) with respect to time variable t , we get

$$s^{-\mu} (1 - \omega s^\rho)^\gamma \bar{u}^*(k, s) - s^{\nu(1-\mu)-1} (1 - \omega s^\rho)^{\gamma\nu} \tag{23}$$

$$\left[E_{\rho,(1-\nu)(1-\mu),\omega,0+}^{-\gamma(1-\nu)} u(k, 0+) \right] = i\eta k \bar{u}^*(k, s) - \zeta |k|^\lambda \bar{u}^*(k, s),$$

where $S[u(k, t); s] = \bar{u}(k, s)$.

Solve equation (23), by using conditions (19)-(20), we get

$$\begin{aligned} \left\{ s^{-\mu} (1 - \omega s^\rho)^\gamma - i\eta k + \zeta |k|^\lambda \right\} \bar{u}^*(k, s) &= s^{\nu(1-\mu)-1} \\ &\quad (1 - \omega s^\rho)^{\gamma\nu} g(k), \\ \Rightarrow \bar{u}^*(k, s) &= \frac{s^{\nu(1-\mu)-1} (1 - \omega s^\rho)^{\gamma\nu}}{\left\{ s^{-\mu} (1 - \omega s^\rho)^\gamma - i\eta k + \zeta |k|^\lambda \right\}} g(k). \end{aligned} \tag{24}$$

On taking inverse Sumudu transform of equation (24), and after little simplification, apply result (17), it gives

$$u^*(k, t) = \sum_{n=0}^\infty \left(i\eta k - \zeta |k|^\lambda \right)^n g(k) t^{\nu(1-\mu)+n\mu-1} E_{\rho,\nu(1-\mu)+n\mu}^{\gamma(n-\nu)} (\omega t^\rho). \tag{25}$$

Taking inverse Fourier transform of (25), get our required result (21).

This completes the proof of the theorem 3.1.

On taking $\eta = 0, \zeta = \frac{i\hbar}{2m}$ in Theorem 3.1, we arrive at:

Corollary 3.2. Consider the following one dimensional space-time Schrödinger equation of fractional order, for a free nature particle of mass m is

$${}_0D_{\rho,\omega,t}^{\gamma,\mu,\nu} (u(x, t)) = \frac{i\hbar}{2m} \Delta^{\frac{\lambda}{2}} (u(x, t)), \tag{26}$$

where $\lambda \in (0, 2], x \in R, t \in R^+, \mu \in (0, 1), \nu \in [0, 1]$, with initial condition

$$E_{\rho,(1-\nu)(1-\mu),\omega,0+}^{-\gamma(1-\nu)} u(x, 0+) = g(x), \quad \gamma, \omega \in R, \rho > 0, \tag{27}$$

and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0, \tag{28}$$

where $\Delta^{\frac{\lambda}{2}}$ is same as we defined earlier and $h = 6.625 \times 10^{-27} \text{ erg s} = 4.21 \times 10^{-21} \text{ Mev s}$ is the Planck constant. Subject to the above constraints, solution of equation (26), is

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) \left(-\frac{ih}{2m}|k|^{\lambda}\right)^n E_{\rho, \nu(1-\mu)+n\mu}^{\gamma(n-\nu)}(\omega t^{\rho}) dk. \tag{29}$$

Proof: For obtaining the solution of Corollary 3.2, we follow same procedure, as we used in the proof of Theorem 3.1, and after little simplification, finally we obtain the desired result (29).

ILLUSTRATION

Example 4.1. To describe solute transport in aquifers, consider the following generalized fractional advection dispersion equation

$${}_0 D_{\rho, \omega, t}^{\gamma, \mu, \nu}(u(x, t)) = -D_x u(x, t) + u' \Delta^{\frac{\lambda}{2}}(u(x, t)), \tag{30}$$

with initial condition

$$E_{\rho, (1-\nu)(1-\mu), \omega, 0+}^{-\gamma(1-\nu)} u(x, 0+) = e^{-x}, \quad 0 < x < 1, \quad t > 0, \tag{31}$$

and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0, \tag{32}$$

where $\mu' = \frac{d}{vL}$ and we consider a dimensionless parameter, called Peclet number, $Pe = \frac{1}{\mu'}$ where L is the packing length. The Peclet number determines the nature of the problem, that is, the Peclet number is low for dispersion-dominated problems and is large for advective dominated problems, d is the dispersion coefficient [$L^2 T^{-1}$] and v' is the Darcy velocity [LT^{-1}].

Our interest is in the solution of (30), for this we follow same procedure, as we applied in the proof of Theorem 3.1, and after little simplification, finally we obtain

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) (ik - \mu'|k|^{\lambda})^n E_{\rho, \nu(1-\mu)+n\mu}^{\gamma(n-\nu)}(\omega t^{\rho}) dk \tag{33}$$

Here $u(x, t)$ represent the analytical expression of solute concentration and $g(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1+ik)} - 1}{1+ik} \right]$.

Example 4.2. Consider the generalized fractional order space-time advection-dispersion equation

$${}_0 D_{\rho, \omega, t}^{\gamma, \mu, \nu}(u(x, t)) = -D_x u(x, t) + u' \Delta^{\frac{\lambda}{2}}(u(x, t)), \tag{34}$$

with the initial condition

$$E_{\rho, (1-\nu)(1-\mu), \omega, 0+}^{-\gamma(1-\nu)} u(x, 0+) = \delta(x), \tag{35}$$

Here $\delta(x)$ is Dirac-delta function and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0, \tag{36}$$

The solution of (34) can be obtained by same technique as we applied in proof of Theorem 3.1

$$u(x, t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} (ik - \mu'|k|^{\lambda})^n E_{\rho, \nu(1-\mu)+n\mu}^{\gamma(n-\nu)}(\omega t^{\rho}) dk. \tag{37}$$

SPECIAL CASES

Some interesting special cases of Theorem 3.1 are enumerated below:

If we set $\gamma = 0$, in (14), then Hilfer-Prabhakar derivative reduces to Hilfer derivative (12), and the Theorem 3.1 reduces to:

(I). Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$${}_0 D_t^{\mu, \nu}(u(x, t)) = -\eta D_x u(x, t) + \zeta \Delta^{\frac{\lambda}{2}}(u(x, t)), \tag{38}$$

where $(0 < \lambda \leq 2)$, $x \in R$, $t \in R^+$, $\mu \in (0, 1)$, $\nu \in [0, 1]$, with initial condition

$$I_{0+}^{(1-\nu)(1-\mu)} u(x, 0+) = g(x), \quad x \in R, \tag{39}$$

and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0. \tag{40}$$

For obtaining the solution of (38), follow same procedure as we used in the proof of theorem 3.1, and use (13), after little simplification, obtain the following

$$u(x, t) = \frac{t^{\nu(1-\mu)+\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) E_{\mu, \nu(1-\mu)+\mu}^1 \left((i\eta k - \zeta|k|^{\lambda}) t^{\mu} \right) dk. \tag{41}$$

Again, use convolution theorem of the Fourier transform to (41), then we get solution of (38), in term of Green's function as

$$u(x, t) = \int_{-\infty}^{\infty} G(x - k, t) g(k) dk.$$

Here Green's function is given as

$$G(x, t) = \frac{t^{\nu(1-\mu)+\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\mu, \nu(1-\mu)+\mu}^1 \left((i\eta k - \zeta|k|^{\lambda}) t^{\mu} \right) dk.$$

If we set $\nu = 1$ in (12), then Hilfer fractional derivative reduces to Caputo fractional derivative operator (10) and the equation (38), yields the following:

(II). Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$${}_0D_t^\mu (u(x, t)) = -\eta D_x u(x, t) + \varsigma \Delta^{\frac{\lambda}{2}} (u(x, t)), \quad (42)$$

where $(0 < \lambda \leq 2)$, $x \in R$, $t \in R^+$, $\mu \in (0, 1)$,
with initial condition

$$u(x, 0+) = g(x), \quad x \in R, \quad (43)$$

and boundary condition

$$\lim_{|x| \rightarrow \infty} u(x, t) = 0, \quad t > 0. \quad (44)$$

For obtaining the solution of (42), follow same procedure as we used in the proof of theorem 3.1, and use (11), after little simplification, obtain the following

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) E_{\mu, 1}^1 \left((i\eta k - \varsigma |k|^\lambda) t^\mu \right) dk. \quad (45)$$

Again, use convolution theorem of the Fourier transform to (45) then we get solution of (42), in term of Green's function as

$$u(x, t) = \int_{-\infty}^{\infty} G(x - k, t) g(k) dk.$$

Here Green's function is given as

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E_{\mu, 1}^1 \left((i\eta k - \varsigma |k|^\lambda) t^\mu \right) dk.$$

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(III). On giving suitable value to the parameters involved in Theorem 3.1, we can obtained same results, earlier given by Haung and Liu [14], Haubold et al. [15], Saxena et al. [16], and Agarwal et al. [17].

CONCLUSION

In this paper, we have presented a solution of generalized space-time fractional advection-dispersion equation. The solution has been developed in terms of Mittag-Leffler function with the help of Sumudu transform and Fourier transform. We can develop the efficient numerical techniques to find solution of various fractional partial differential equations arising in various fields by considering these analytic solutions as base. For future research, the methodology presented in this paper can serve as a good working template to solve any fractional advection-dispersion equations in higher dimensions.

AUTHOR CONTRIBUTIONS

VG, JS, and YS designed the study, developed the methodology, collected the data, performed the analysis, and wrote the manuscript.

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