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How mathematicians characterize and attempt to develop understanding of concepts and definitions in proof-based courses

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Mathematics education research has long focused on students' conceptual understanding, including highlighting conceptions viewed as problematic and looking for ways to develop more desirable conceptions. Nevertheless, limited research has examined how mathematicians characterize understanding of concepts and definitions or promote activities beneficial for students. Based on interviews with 13 mathematicians, we present thematic characterizations of what it means to understand a concept and definition, highlight activities mathematicians believe assist students' learning, and examine their reasons for promoting these activities. Results include mathematically grounded descriptions of what it means to understand a concept but general descriptions of approaching and supporting learning. Implications include a need for attending to intended meanings for "understanding" in context and how this impacts appropriate activities for developing understanding, as well as a careful examination of the extant research literature's claims about seemingly unified notions of conceptual understanding.

KEYWORDS

conceptual understanding, definition understanding, teacher beliefs, instructional practice, proof, interviews, thematic analysis

1 Introduction

Definitions are considered foundational across science (c.f., Butt and Royle, 1980; Gillespie and Giardino, 1998; Zukswert et al., 2019), mathematics (Edwards and Ward, 2008), and social science (c.f., DiRenzo, 1966) for clear and precise communication. Butt and Royle (1980, p. 29) described the possible danger to scientific progress if future generations "inherit a language that is clouded with ambiguity and vagueness," while Gillespie and Giardino (1998, p. 427) claimed "without precision and consistency, the information may not be understandable to others, or it may be misleading." Because of the importance of definitions for communication in STEM disciplines, there have been numerous studies that are reasonably characterized as exploring student challenges with definitions in the sciences and mathematics. For example, Park and Choi (2013) explored student difficulties with chemistry concepts that involved mathematical representations, describing South Korean students' challenges with manipulating the definitions and the ideas. Mathematicians and mathematics educators agree that being able to state and use definitions is essential for mathematical thinking, conjecturing, and proving (Edwards and Ward, 2008; Alcock and Inglis, 2009; Alcock and Weber, 2010). A definition often provides the structure of the proof or the assumptions to work with (Selden and Selden, 2009; Weber, 2010). Developing conceptual understanding is also an important goal in advanced undergraduate coursework (Weber, 2001; Weber and Alcock, 2004; Fukawa-Connelly and Newton, 2014). For example, Weber (2013) noted that a body of early research suggested students also fail to learn the "intuitive notions" that mathematics educators believed desirable for students to learn (e.g., Leron et al., 1995; Weber, 2001; Weber and Alcock, 2004). Since then, there have been explorations of students' difficulties with mathematical definitions, such as a study of South African students' difficulties with definitions in set theory of Shaker and Berger (2016), which focused on both misunderstanding definitions and the inability to successfully use them in proofs. Haj-Yahya (2022) explored how Arabic high-school students in Israel understood the characteristics of mathematical definitions and claimed that a misunderstanding led to an inability to interpret theorems and unpack their logical structure. Noto et al. (2019) carried out a similar study with preservice teachers in Indonesia, although in the context of geometry, with similar findings. Moreover, there have been numerous attempts to improve student understanding of definitions in science (c.f., Zukswert et al., 2019) and mathematics (Zazkis and Leikin, 2008; Larsen, 2013) and to improve students' ability to use mathematical definitions in proof-writing (c.f., Jordan, 2019; Kempen and Biehler, 2019; Valenta and Enge, 2022). Yet, for all of the mathematics-educator-developed interventions to improve student understanding of definitions, we were unable to find any exploration of what mathematicians mean by student understanding of definitions and concepts. That is, we appear to be attempting to solve a problem for instructors without fully understanding the task–to build tools in support of student understanding, we need to know what mathematicians desire for and from their students.

Weber (2001, p. 691) argued that "competent performance in abstract algebra might require conceptual understanding beyond being able to interpret the definition of concepts." That is, while definitions specify the concept, mathematics education literature has consistently treated conceptual understanding as different from understanding the definition, and both appear to be important for success in proof-based mathematics. While there is a lack of research on what mathematicians believe it means for students to understand a definition and concept, mathematics educators have developed a number of theoretical tools to explore these ideas. Perhaps the most used in exploring student understanding of definitions in undergraduate mathematics are the constructs of concept definition and concept image of Tall and Vinner (1981). Tall and Vinner distinguished between the statement of a definition, the *concept definition*, and the other ways of thinking that are useful or brought to mind when the concept is considered, the *concept image*. Numerous researchers have noted that examples of a concept should form an important part of the concept image. Watson and Mason (2005) argued that it was critical the students should be able to construct their own examples. Fukawa-Connelly and Newton (2014) used this as a tool to analyze the teaching of a group-theory course (which, in the US context, is a fourth-year university course). They explored the examples the professor gave of a group and how the collection of examples provided certain opportunities to learn the concept but might not have illustrated all aspects of the definition. Vinner and Dreyfus (1989), among thousands of others, have used the ideas to analyze student thinking, where they explored ways of understanding the concept of function that high school and university students held. They found significant differences between the definition of function and the concept images that students held. Moreover, these ideas have widespread use and intuitive appeal; for example, they resonate with Thurston (1994) claims about what it means to understand a concept.

Nevertheless, we are unaware of a body of research that attempts to capture teaching mathematicians' meanings for student understanding of mathematical concepts and definitions or even whether those are meaningfully distinct. Thus, this study explores mathematicians' thinking about what it means for students to understand mathematical definitions and concepts, the ways that they claim to promote this understanding, and their thinking about the work that students should do outside of class to develop understanding. We draw on the notions of concept definition and concept image in doing so.

To lay the groundwork for this study, we first highlight prior work on conceptual understanding and instruction in proof-based mathematics courses. In particular, instructors often model their own mathematical activity (Fukawa-Connelly, 2012; Fukawa-Connelly et al., 2017; Pinto, 2019), typically by providing running commentary of their thinking as they teach (Artemeva and Fox, 2011). This running commentary frequently consists of formal statements (reading what is written on the board) and informal ways of thinking, both of the content and meta-commentary on how they do mathematics. Fukawa-Connelly et al. (2017) provided corroboration of these basic claims and noted that examples and informal ways of understanding content were common parts of the presentation of advanced mathematics courses, including explanation of concepts and examples. Pinto (2019) and Fukawa-Connelly (2012) both describe the modeling of mathematical practices as common parts of lecture. That is, mathematicians typically present a formal definition of a concept, a number of examples, as well as providing informal ways of thinking about the concept (e.g., they provide a *formal concept definition* as well as ideas that might become part of a student's *concept image* during their lectures). Viirman (2015) explored the lecturing practices of university mathematics faculty teaching calculus and linear algebra and argued that the participants used multiple representations of concepts as a means of conveying multiple ways of thinking about any particular concept (e.g., provided multiple aspects of a possible *concept image*). All of these might be understood as exploring how mathematicians present definitions as well as additional ideas related to mathematical concepts. There has been significant research exploring how mathematicians think about teaching, especially about the presentation of theorems, and explorations of how mathematics is presented in lecture (see Melhuish et al. (2022) for a recent review). Yet these studies have only seldom explored what the mathematicians want their students to learn from these presentations of definitions and other mathematics, and have typically done so in the context of a single definition.

Reading across literature on students' understandings of definitions and concepts and on collegiate mathematics instruction, we see a need to explore three broad questions:

1. What are (some) ways that mathematicians characterize what it means for a learner to understand a definition of a concept? That is, what, if any, are ways that mathematicians distinguish between understanding a definition of a mathematical term vs. understanding a mathematical concept?

2. What are activities that mathematicians want students to take outside of lecture in order to develop these types of understandings?
3. Why do mathematicians believe the desired activities support students in developing the desired understandings?

We entered the study assuming that mathematicians would distinguish between knowing the statement of a definition and understanding a definition. However, we tried not to make assumptions about their meanings for understanding the definition. In particular, we did not assume that they held mental models for student understanding similar to those described in the mathematics education literature. Rather, our goal was to examine their characterizations in order to compare them with those of the mathematics education literature as well as frame practical discussions between mathematicians and mathematics educators about students' understandings and activities to support developing students' understandings.

2 Theory and literature

2.1 Theoretical perspectives

2.1.1 Theory of reflective practice

We adopt the theory of reflective practice of Schön (1983) as our epistemological perspective. Schön characterizes professionals' knowledge as something demonstrated and considered through action, even if that knowledge cannot always be fully articulated. In so doing, he rejects the positivist premise that all knowledge must be an application of theory; rather, he suggests that practitioners have knowledge of what works in their practice that may not yet be aligned to a general theory. In particular, he distinguishes among participants' knowing-in-action, reflecting-in-action, and reflecting-in-practice. Knowing-in-action refers to spontaneous actions that do not need to be thought about before being done. Reflecting-in-action (or reflection-in-action) refers to a professional's consideration of their own actions, potentially while engaged in that action, such as reexamining an activity that went well to determine the aspects that went well. "Thinking on your feet" and "learning by doing" are emblematic of such reflection-in-action (Schön, 1983, p. 54). Of note, Schön claims, "Because professionalism is still mainly identified with technical expertise, reflection-in-action is not generally accepted–even by those who do it–as a legitimate form of professional knowing" (Schön, 1983, p. 69). Thus, this component of the theory is especially applicable to our work and highlights that professionals (mathematicians) have knowledge specific to teaching mathematics in addition to their content expertise, even if they do not view themselves as having such specialized knowledge. Reflecting-in-practice refers to practitioners' analysis of their own understanding, and might be viewed as the activity in which they engaged via the interviews themselves. Thus, grounded in the theory of reflective practice of Schön (1983), we adopt the stance that mathematicians have valuable applied knowledge of teaching.

While much of Schön's work emphasizes the reflectiveness and worth of practitioners' ways of knowing and acting, he also permits a role for outside observers like researchers. In particular, researchers might be viewed as theorizing about practitioners' ways of knowing in a way that moves knowing-in-action to knowledge-in-action (i.e., translating an activity or process into an object or body of knowledge that can be described and analyzed). This is done by articulating a theorized mechanism or understanding that can be tested. We incorporate this lens by observing that the aspects of practice to which mathematicians attend are important to document, even though we, as researchers, then seek to complement mathematicians' noticings with language to permit theory building. However, we do not find it productive to distinguish between (unwarranted) beliefs, justified or warranted beliefs, and knowledge in this context because in faculty's decision-making they operate the same way: shaping intended outcomes and instruction. Because we asked participants to explain what it means for students to "understand a concept" and "understand a definition of that concept," participants' responses largely (though not always) treated concepts as topics that might be addressed in a course, whether a fixed but unspecified topic or a specified topic like compact sets. Thus, we use the term *concept* in a topic sense in this paper.

2.1.2 Concept image and concept definition

We also utilize the theoretical framing of concept definition and concept image of Tall and Vinner (1981). Tall and Vinner distinguished between the statement of a definition, the *concept definition*, and the other ways of thinking that are useful or brought to mind when the concept is considered, the *concept image*. For instance, the concept of function could include a formal definition accepted by the mathematical community [e.g., A function from set S to set T is a subset F of $S \times T$ such that for each element s in S, there is exactly one element t in T such that (s, t) is in F.], as well as associated concepts like graphs of functions, particular examples of functions, and ways of using functions.

The concept image and concept definition constructs were developed first as theoretical categories, perhaps grounded in intuitive notions, and then used by mathematics educators as analytical tools. Tall and Vinner wrote:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures (p. 152).

They further described the portion of the concept image that was brought to mind in any given mathematical situation as the *evoked concept image*. They also distinguished between the definition provided in the lecture or text, calling it the *formal concept definition*, while a definition that someone develops using their own language is a *personal concept definition*. While this theoretical perspective has been used extensively to analyze student understanding in undergraduate mathematics education research, we here apply it to examine whether mathematicians' characterizations of understanding in fact align with this common framing of the research community.

2.2 Literature

We also draw on extensive literature characterizing mathematicians' practices for their own understanding and in the classroom.

2.2.1 Mathematicians' practices for their own understanding

The first strand of literature we draw on is the notion of mathematicians as enquirers (c.f., Burton, 2004). In that vein, Parameswaran (2010) explored ways that mathematicians attempt to learn a new concept or definition via interviews of five mathematicians and surveys of 12 (including those who were interviewed). The findings included four broad processes: studying examples, using the new definition to prove theorems, exploring equivalent definitions, and encountering and resolving cognitive conflicts. Parameswaran's participants appeared to engage in both mechanical manipulations and reasoning focused on properties of objects as a means of developing understanding of a new definition. Wilkerson-Jerde and Wilensky (2011) explored processes that mathematicians use to develop understanding of an unfamiliar proof and claimed that they refer to definitions, reason with examples, and engage in self-questioning, typically using combinations of these processes simultaneously. Burton (2004) similarly described examples as key tools that mathematicians use to develop understanding of new ideas. This literature, on how mathematicians seek to understand, suggests that they value informal ways of thinking about concepts, examples, and alternative forms of definitions and statements (all of these might be understood as aspects of the concept image).

2.2.2 Observations of classroom practices regarding definitions

The second strand of literature we draw on is that of mathematicians presenting content. While there is significant literature exploring classroom presentations of proof (c.f., Melhuish et al., 2022), there is relatively little documentation about classroom teaching practices regarding definitions. The study of Pinto (2019) explored how two mathematicians presented "the same" lesson about the formal definition of derivative in a real analysis course. Both introduced the formal definition, stated a theorem, and showed several examples and some applications but did so differently. Pinto argued that one, Yoav, carefully stated the meaning of each term in the definition while the other, Amit, used informal, metaphorical, language. Pinto argued that Yoav and Amit modeled different ways that students should develop understanding of definitions when they encounter them. Moreover, their rationales were also quite different, with Yoav claiming that being able to state the definition of terms was important while Amit claimed that mathematicians learn new definitions by "playing" with them and instantiating them visually and via metaphor. Pinto argued that both were modeling ways of coming to understand definitions that they found valuable.

Some other studies have explored classroom presentations of mathematical content that also included the presentation of definitions in proof-based courses. Artemeva and Fox (2011), Fukawa-Connelly et al. (2017), Fukawa-Connelly and Newton (2014), and Paoletti et al. (2018) all described definition presentation and suggest broad commonalities in presentation. Broadly, the construct of "chalk talk" includes presenting mathematical content along with narration of more informal ways of thinking and meta-mathematical claims (Artemeva and Fox, 2011). The study of Fukawa-Connelly et al. (2017) of 11 proof-based lecturers suggested that these informal ways of thinking are common and include both examples (typically written) and informal and metaphorical ways of understanding content (typically said aloud and not written). Some participants included informal ways of thinking about concepts for which they provided formal definitions, typically stated aloud, sometimes with diagrams or other drawings on the board. This meta-mathematical narration included mathematical processes such as "why is this a sensible name for a variable" and heuristics for accomplishing mathematical tasks. These can be understood as attempts to convey practices and ways of thinking that are mathematically productive. When the goal is to "avoid confusion" or "make sensible choices" there is no mathematically necessary reason for a particular choice (e.g., all variables could be named x, all with different subscripts). Fukawa-Connelly and Newton (2014) also described the installation of a definition, but then focused on the exemplification that followed. They noted that immediately following the statement of the definition, the instructor gave both examples and a non-example (that was then modified into an example). Finally, Paoletti et al. (2018) listed numerous examples of questioning, one of which focused on the installation of a definition in an abstract algebra class and included meta-mathematical aspects in which the instructor's questions were focused on "what's new here?" in moving from the definition of a sub-ring to an ideal and how the definition of ideal is different from that of sub-ring. These explorations typically focus on what Tall and Vinner (1981) would call the concept image. We also highlight Pinto's (2019) investigation into two Teaching Assistants' presentations of a definition in a calculus class. One was focused on the meaning of terms and symbols, carefully explaining each. The second gave a presentation that used graphs and intuitive and informal explanations of the concept. Thus, we might understand mathematicians as presenting both formal and informal ways of understanding a particular concept, or a formal concept definition and multiple aspects of a concept image.

2.2.3 Interview studies on mathematicians' beliefs, goals, norms, and values

The third strand of literature we draw upon focuses on interviews with mathematicians about their teaching in advanced mathematics (e.g., Nardi, 2007; Biza et al., 2014) although typically with a focus on the teaching of proof (e.g., Alcock, 2010; Hemmi, 2010; Weber, 2012; Lai and Weber, 2014; Cook and Fukawa-Connelly, 2015; Lew et al., 2016; Woods and Weber, 2020). Generally, these interview studies present the participants' goals for instruction and claims about their pedagogical actions and analyze the relationship between them. The interviewed mathematicians in these studies tended to believe that understanding advanced mathematics involves acquiring informal understandings of the content, citing the importance of exemplifying (e.g., Alcock, 2010) and having visual or kinesthetic representations of important mathematical ideas (Weber, 2004; Lew et al., 2016). As one such example, Iannone and Nardi (2005) conducted focus groups with 20 faculty members from United Kingdom universities in which they explored questions of pedagogy, and Nardi (2007) drew on these interviews in creating a "composite mathematician" in conversation with a mathematics educator. We characterize the findings from both as that the mathematicians were generally of the belief that students learn best with interaction and that students learn best when moving from the concrete to abstract in terms of representations. These beliefs generally align with the claimed practice of mathematicians (c.f., Katz et al., 2018) while there is also evidence that mathematicians think about the examples that are "most iconic" for a given definition, and that they would expect students to be able to call on (Cook and

Fukawa-Connelly, 2015). However, mathematicians also recognize that not all values upheld by definitions and espoused by mathematicians are portrayed to students–whereas clarity in and for communication is emphasized through instruction with definitions, freedom of choice in the use and creation of definitions is not typically emphasized in instruction (Rupnow and Randazzo, 2023).

Fundamentally, mathematicians engage in their daily instructional practice based on their experience as learners and teachers–they are teaching a specific audience in a specific subject and consider what they consider to be "normal" classroom activity for the course in designing their lessons. For example, there is a "convergence" around the "right" number of homework problems to assign in a proof-based class (c.f., Rupnow et al., 2021), that balances the desire to promote student understanding with their willingness and ability to complete the work. Similarly, the genre of chalk-talk is remarkably similar across cultures and countries, and within the United States lectures are generally very similar in terms of structure, pacing, and the types of content and pedagogical practices engaged (c.f., Artemeva and Fox, 2011; Fukawa-Connelly et al., 2017; Johnson et al., 2018). That is, while there are differences, there is also a cultural convergence in which there are significant similarities in practice. We further problematize the notion of cultural expectations in exploring what mathematicians intend by "what it means to understand a definition" and explore behaviors they would expect students to engage in as a means to develop that type of understanding. We suggest that these behaviors are a form of "folk wisdom" passed down via practice and observation of mentors and other scholars [c.f., claims of Johnson et al. (2018) about sources of insight for mathematicians in their teaching]. Moreover, these sets of behaviors and practices are believed to be effective based on the "accumulated wisdom of past practice" rather than studies of their efficacy.

3 Methods

The goals of this study were to investigate mathematicians' thinking about what it means for a student to understand mathematical concepts and productive means for students to acquire that type of understanding. Thus, we drew on reflective interviews in which the participants were asked to describe their thinking in the abstract and to instantiate it in their teaching experience. Our goal is to (1) present accounts of what mathematicians believe it means to understand a concept; (2) present accounts of what mathematicians believe are beneficial activities for students to develop desirable understandings; and (3) present accounts of why mathematicians believe these activities are beneficial. When mathematicians made claims about a behavior being beneficial for students, we assumed they had beliefs that support their claim, even if implicit. As much as possible, we asked them to explain why they believed the behaviors were beneficial, but often we were left to infer beneficialness based on their claims about what it means to understand a concept or definition.

3.1 Participants

We recruited 13 participants from four different masters or doctoral universities across the United States. All of the participants had recently taught a class that included students writing proofs but had a description that included learning specific mathematical subjects. Course content included abstract algebra, linear algebra, real analysis, stochastic processes, optimization, and topology. We did not ask the participants their gender identity or years of experience and use gender-neutral pseudonyms for our participants throughout.

3.2 Data collection

One participant was interviewed in-person and one via phone by the first author; the remaining 11 participants were interviewed via Zoom by the second author. Interviews ranged from approximately 25 min to more than 60 min. All of the interviews were recorded, and we then generated transcripts (first via AI transcription, then human correction).

We followed a semi-scripted interview format (Fylan, 2005) in which we asked participants general questions about their beliefs, goals, and practices about the teaching and learning of definitions and concepts and then asked them to instantiate with examples from recent courses. In cases where they described non-proof-based courses, we attempted to redirect their instantiation. Similarly, if they did not instantiate, we prompted them to do so. Our questions explored what they believe it means for a student to understand a concept, how that is related to understanding a definition of that concept, actions they believe students should take to develop the desired understandings, and what they do to promote students engaging in those actions. We did not provide participants with a definition of "concept." We also specifically asked whether students need to know the statement of the definition presented in class and asked for a justification for the response. We followed up with requests for more information as appropriate.

These questions, which especially focus on participants' actions and actions they desire students to take, are fully grounded in the theory of reflective practice of Schön (1983). In particular, participants were encouraged to reflect on their actions and were not required to be able to fully articulate why they do those things, though any reflections on why those actions were important were encouraged. This data collection then supports the theorizing of the researchers.

3.3 Data analysis

For claims about understanding of concepts and supports for understanding, we drew on reflexive thematic analysis (Braun et al., 2019). Thematic analysis has been used in the mathematics education literature to describe a number of different types of qualitative analysis focused on researchers generating themes (patterns) from data based on repeated, systematic analysis (coding data). Furthermore, the research team is acknowledged as bringing prior knowledge and experiences to the analysis as opposed to being a "blank slate." Reflexive thematic analysis is a particular type of thematic analysis in which themes result from the analytic work of the researchers and are not domain summaries predetermined before analysis. Moreover, the researchers are positioned as storytellers who interpret the data in light of their own positionalities.

This methodology permitted building on prior research on understanding as we coded. Specifically, Pinto (2019) and Parameswaran (2010) suggested that mathematicians might hold multiple interpretations for what understanding a definition means; thus, we began with initial codes for understanding that drew on their work. These included important markers of understanding, such as being able to state the definition for a concept and the definitions of all terms used in that concept's definition, being able to formulate equivalent definitions by drawing on conceptual understanding, and using the new concept in theorems and proofs. Because of the importance ascribed to examples in mathematics education literature, we further coded any claims that the mathematicians made about examples, including non-examples, as indicating understanding of the concept. We then came to consensus on the markers of understanding present in each paragraph of text, though we ultimately report our participants' characterizations by person (i.e., our unit of analysis for coding was the paragraph/uninterrupted response to a prompt, but our reporting unit of analysis is the participant).

While the literature suggested potential definitions of knowing and understanding, in keeping with Schön, we view one contribution of this study as the theoretical claims about how mathematicians characterize what it means to know and understand a concept in proof-based mathematics. Hence we argue that using the bottom-up defining process that results in the descriptions below is the appropriate approach to the development of our themes. Nevertheless, we observed that our participants' responses largely aligned with emphases on the definition itself (concept definition) and related important notions for understanding a concept (concept image). To that end, we re-examined our data. This led us to group characterizations of understanding into categories aligned with the concept image and concept definition and also to group beneficial activities into categories. Thus, throughout the process, we employed multiple rounds of coding (Anfara et al., 2002) and used our consensus-building coding comparisons to build confidence in the validity and reliability of our results.

4 Results

We respond to the research questions by exploring how participants characterize understanding a concept with respect to the definition specifically and with respect to other facets. These results are presented in two categories about understanding that synthesize the participants' claims. We then characterize three types of activity that participants viewed as likely to support students' learning. Finally, we examine mathematicians' rationales for why these activities and supports would be valuable. We view participants' responses as examples of prompted reflecting-in-practice.

4.1 Characterizing understanding

Understanding a concept includes being able to state a definition, but it goes beyond that. The mathematicians used language similar to notions of concept definition and concept image of Tall and Vinner (1981). Participants differentiated between definitions and concepts, such as River's claim:

> "And the formal definition [of a convex function] is kind of just like, this function satisfies this inequality, from from an algebraic standpoint, and then, you know, we'll probably acknowledge that this seems somewhat abstract and… draw the picture, which is basically like, 'Oh, well, if I take any chord with this graph, that chord is going to be above the graph.' And hopefully, kind of pinned down that, that kind of idea that, that probably shouldn't take more than five or 10 minutes, and then we'll kind of get into probably a few different kinds of convexity tests beyond just that concept. So, you know, we can say, 'Oh, well, if it's smooth, then we can take the tangent line, that's going to be below the curve', and then maybe work out a kind of sketch of why that's true."

We interpreted River as suggesting that the formal definition is a collection of words that specifies a set of conditions that, in this case, a convex function, must fulfill. River notes that the definition "seems somewhat abstract" and then suggests that drawing pictures and additional tests for convexity help students develop understanding of the concept. That is, a mathematical definition might capture one particular sense of an idea, but the concept typically includes multiple ways of thinking about that same idea. Similarly, Winter claimed:

> "When I say definition, I would like it to mean very specific. And if it's a math class, it should be very specific and no ambiguity. No self contradiction going on whatsoever. And, if I define a terminology, it should exist, like it shouldn't be an empty set… but when I say concept in mind, it's a concept in my class, it's just a word that is more ambiguous like I will write a huge circle and… Yeah. So, it will be… it can be more ambiguous and more… heuristic?"

We interpreted Winter as claiming that the definition is a rigorous statement. Winter's claims that a definition should be "very specific" and allow "no ambiguity" are consistent with mathematical norms (e.g., Rupnow and Randazzo, 2023). In contrast, Winter noted that a concept is "more ambiguous and more… heuristic," which we interpreted as meaning a concept encompasses the definition but also includes more, and that it may not be possible to specify the limits of a concept. Winter's claim is also sensible in relation to mathematical norms–a definition has precise terminology and does not allow ambiguity, while more freedom of choice is permitted in creating a concept (Rupnow and Randazzo, 2023). In the sections that follow, we further characterize how our participants characterized knowing a definition, in alignment with possessing a concept definition, and how they characterized understanding a concept, in alignment with characterizing a student's concept image.

4.1.1 Understanding a concept means possessing a valid concept definition

All of the participants indicated that they believed knowing a statement of the definition was an important component of understanding a concept, which might be viewed as centering students' possession of a concept definition. There were two ways the participants described "knowing" a definition. One way focused on being able to recite a memorized version of a definition (i.e., memorizing a formal concept definition). The other focused on being able to recreate a viable definition (i.e., creating a viable personal concept definition).

4.1.1.1 Concept definitions can be memorized word for word

Eight of the 13 participants made at least one claim we interpreted as indicating that being able to state the definition of a concept as presented in class (i.e., word for word) is a marker of understanding the concept. Noel, who taught abstract algebra, claimed:

> "I kind of think of that memory stuff as like a zero measure of success. Math is hard, and sometimes the best you can realistically be expected to do is state a definition from memory. And if that's where you're at, then that's where you're at. Go with it. You don't want to put that down, but you want to view it as a starting point toward a better understanding."

We interpreted Noel's claim as indicating that being able to state the definition from memory is a form, albeit minimal, of understanding.

Xylon claimed that proficiency with proof and the development of subsequent material requires being able to state definitions for foundational concepts, though being able to state specialized definitions would be less important:

> "I think that for the foundational concepts you encounter, it's absolutely important. Yeah, if a student doesn't know what a group is, and keeps having to go back to the book and consult to figure out what a group is, I think it's just going to slow them down… I have observed that, you know, students have that kind of stuff under their fingers. It doesn't interrupt their thought patterns to have to go back. And, you know, on the other hand, I mean, there's some fairly esoteric definitions, like a non-separable field extension or something like that, where if you gotta run back and and see what that means, that's okay. So I guess I don't have one fixed answer for your question."

We note that Xylon is differentiating between "foundational" and "esoteric" concepts as a means of thinking about which definitions students should be able to state.

4.1.1.2 Concept definitions can be re-created

In contrast, 10 of the participants (five overlapping with the "word for word" group above) valued being able to extract a logically equivalent definition from their concept image (Tall and Vinner, 1981). Taylor made specific claims about the importance of definitions in mathematics, contextualized in analysis, explaining:

> "It got me to start trying to make clearer to people that this is a difference, that if you haven't seen a proof course before, then the fact that we define a function to be continuous at seven if blah, blah, bah, bah, bah, that's there isn't anything else behind the curtain. There's no curtain. It's all, it's all upfront. That's what we mean. We don't mean anything else. We've chosen to declare anything that satisfies that to be continuous."

We note that Taylor emphasizes that mathematical definitions are stipulated, meaning these definitions create concepts (Edwards and Ward, 2004, 2008), though not all people are aware of this aspect of mathematical definitions. At the same time, Taylor does not believe that it is useful to memorize definitions; instead, Taylor believes in re-creating them:

> "I hate word for word. I, I mean, historically, in, in my experience of mathematics, teaching, people who insist on word for word. I mean, yeah, um, I don't see that memorization of any form is valuable in mathematics. But that's a very personal thing. And it's based on the fact that my memory is useless. And I, so I phrase this to my students as I am an advocate of zero memory mathematics. But what that means is, I do a lot of mathematical re-creation, as I'm going along."

For Taylor however, even as an advocate of "zero memory mathematics," understanding means knowing what the definition says, being able to give a correct definition, and being able to use the definition in writing proofs:

> "But they've got to know what the definition says, they've got to be able to tell me a correct definition of what continuity is, otherwise, they cannot do anything. And I will tell you, yeah. I want them to be able to state things. I just don't have any interest in the form of the statement other than its correctness."

Taylor has a clear goal for students: that they do not memorize the definitions of concepts, but that they are able to state a correct definition.

4.1.2 Understanding a concept means having a robust concept image

4.1.2.1 Concept images include examples and/or non-examples of the concept

All participants highlighted a role for examples in understanding a concept. Eight participants claimed that understanding a concept involves being able to state examples and/or non-examples of the concept. Skylar explained that for a student to demonstrate understanding, "I would want a student to be able to give me a bunch of examples of things that illustrate the concept. And then also give me a bunch of examples of things that…do not fall into that concept or definition." Xylon made a very similar claim, "Well, I guess it varies, depending on what the concept is… So I think, you know, to know something means…you got to be able to give an example of something, and not such a thing." Both Skylar and Xylon made explicit claims that knowing a concept means being able to give both examples and non-examples of the concept.

Similarly, five of our participants believed that one should be able to generate new examples and understand that multiple objects will likely satisfy an abstract definition. Misha believed some understanding is demonstrated by being able to determine whether or not a particular instance is an example of a concept, but a greater understanding requires having access to more than a single reference example:

> "I would say someone understands a definition if you say, 'Okay, is this one of those?' And if they can competently answer that, then I guess they would understand the definition. But why, right? That's the big thing…If [the integers is] the only example you can come up with for a commutative ring, then I'm thinking maybe

> you don't really know very much about commutative rings. Because if you can't come up with another example, then you sort of missed the point, right?"

Seven of our participants expected students to not only be able to determine example status, but to be able to justify that claim. Quinn highlighted how justification requires more thought than simple verification of examplehood:

> "I mean, when you have a, like, there are abstract concepts, like, um, equivalence relation, equivalence relation, it's something that students struggle with. Yeah. And you try to give as many examples, you know, two people are equivalent if they are born in the same state.….And I find that it's useful sometimes to give examples and you don't tell them the answers, you force them to actually think about it, because if I give you the answer, I say, okay, we're going to prove that this is an equivalence relation. Now, if you're going to prove that we know it's true. So let's just go on, go on automatic pilot. … But if we are asked, okay, wake up and think about this. Is this reflexive or is this then that, uh, again, I think it's important for them to, again, to have, make a conscious effort and think about it."

By setting up a question where students have to determine which tactics to use (here, possibly look for a counterexample or seek to prove the statement in general), students need to engage with the task more thoughtfully than if told which solution path to take.

4.1.2.2 Concept images include mechanisms for determining when to use the concept to solve problems

Eight participants claimed that for students to understand a concept means they can identify when to use it in solving problems, which we view as a facet of students' concept image. Some claimed that students should be able to explain when and why to use the concept (e.g., to explain how it is helpful). Oakley suggested that understanding a concept includes:

> "When you want to solve a problem and you immediately know… Yeah, you don't have to be given that "use whatever theorem to prove." The minute you see the problem, you know, immediately that aha, this, this calls for me to use that theory here."

For Oakley, understanding a concept requires that students quickly recognize whether that concept would be useful to solve a problem, such as writing a proof.

Urban gave a different formulation, but one that also indicates understanding requires recognizing that one should use the concept in solving problems, claiming, "I want them to explain in words, you know, how it applies in these situations, and I can use it to solve a problem in a novel circumstance, right." In this case, Urban goes beyond the idea of having intuition and instead wants students to be able to explicitly describe how the idea is useful in a given problem.

Taylor claimed that to understand a concept means that "people will [be] light on their feet, that they have the ability to, sort of, like, bring in different things, to make sense of what they are talking about." Taylor then provided instantiation in the context of linear algebra:

> "Now, because especially in linear algebra, that feels crucial, because, you know, well, there's the ideas of geometry, there's the ideas of solving linear equations, there's the ideas of linear transformations. And they all come together around this core idea of a matrix that, that then if you are not willing to acknowledge the geometry, or you're not willing to acknowledge the linear equations aspect, it's all, it's very difficult to have a real understanding of, of what's going on."

We interpreted Taylor as claiming that understanding a concept requires being able to productively relate and draw on multiple ways of thinking about the concept. Taylor's instantiation described matrices in terms of geometry and solutions to systems, and both are needed in order to "understand what's going on" with the concept of a matrix.

Taylor also described understanding in terms of metacognition:

> "I do feel that one of the skills that is crucial for people to develop understanding and is underappreciated as a, as an aspect of understanding by students is that they've got to know what they know and what they don't know. And, and I tried to describe the extent to which I'm constantly checking what I am learning against what I know. And without a strong collection of things to check against. I don't see how you would develop this light on your feet, being able to bring in other things, kind of state that I'm aiming for."

Oakley characterized understanding a definition in terms of ways that knowledge could be applied:

> "When you understand the definition… you understand the definition, but, but then you have to really see examples that show different perspectives of that definition. And then you. You put it in context, like with, uh, with the things that you knew before… so then it connects with the things that, so it's attached, it puts in perspective of. That everything that you, you have built so far. So that is, I would say. And then when you… want to solve a problem and you immediately know, now I have to use I don't know or whatever that's that, that I would say that you understood, as a student, something. Yeah."

In the above, Oakley focuses on multiple aspects of understanding. While Oakley mentions "knowing the definition," this is only a first step in the process of coming to understand a concept. Oakley then suggests that examples help a learner understand different aspects or perspectives of the concept. Similar to Taylor, Oakley claimed that understanding requires connecting the concept to other, related ideas. Finally, Oakley claimed that students know when to use the concept in solving a problem.

Noel characterized manners of speaking as a form of understanding:

> "They speak in complete sentences, first of all, and not contrived ones at that. They ask about things that are not directly pertinent, or that are not exactly verbatim what has already been said. They anticipate the next thing that we're planning to talk about in class. But maybe simpler than that, rather than the question of the form, 'How do you do it?' they ask things like, 'Is the following correct?

> Is the following specific thing right?' They offer something in the process of asking their question as opposed to just demanding something from you."

Much like Oakley, Noel claimed that students could indicate understanding in their question-asking, but provided the additional idea that they could do so by including specific ideas rather than "demanding something from you." Noel's focus for understanding was on how students speak and write, and repeated the idea that understanding is nuanced:

> "So I think for me, how do I sort students into those who understand and those who don't? Well, it's a broad spectrum of course and… more than other people I probably value how students speak and write more than I value how they perform on tests, particularly tests that sort of just ask for rote calculations… Better, I do often put true/false questions on my test. And I think those do a much better job of testing understanding than sort of calculations that students can do by mimicking an example that they looked at the day before the test."

Here, Noel reiterated the idea that understanding is not a binary but rather a spectrum, and that calculations on exams are not a good means of gauging student understanding. Similarly, Quinn said that "there are levels of understanding" and characterized mechanical manipulation as a surface level understanding while knowing how and when to use the concept is a deeper level understanding.

4.2 Activities and supports for understanding

4.2.1 Do the homework

Ten of our participants explicitly highlighted the importance of doing homework to gain understanding. Taylor claimed students should do the homework as a means of developing their conceptual understanding, explaining, "Do the homework. Just because, I mean, you know, that's, that's what the homework is there for. It's, if it's well written, it's there to show you around the world of continuity a little bit." Taylor continued by explaining why the homework is valuable:

> "So it's practice. Thinking about the ideas…like, practicing free throws that anybody can do it. You know, if you just practice enough, I mean, I'm not gonna be an exception… so the problems are there, to make you feel at home in that little world of ideas. And, … without living there a bit, you're never going to feel at home. That's an awfully wishy washy way to describe what it is, but that, that's how it feels to me that you've got to live."

Taylor makes two specific claims about the activity of doing homework–that students should get a sense of the concept by doing it, and that it is a form of practice. Taylor suggests that the homework is "to make you feel at home in that little world of ideas" and that to feel at home requires "living there a bit." We equate this with a notion of "spending time thinking" about the ideas. Thus, we interpret Taylor as suggesting that the homework introduces ways of thinking and supports student engagement via time and conceptual effort. At the same time, Taylor suggests that in analysis, the problems appear formulaic:

> "When I look at the problems in [analysis], is that well, the proof writes itself, because, you know, everything up to the key idea is forced on me. And now I've got to get over this one key obstacle, and how am I going to do that, and sometimes that's harder than it is at other times… trying to, trying to clarify for students that, you know, part of what I'm teaching them is not routine, but routine for people for this course. And part of it is that that's why they're on 78 almost identical problems, where everything is routine, there are theorems that have one key idea, and everything else around it is okay, so now we're going to get to this idea but, but, but now we don't ever do that again. Now we don't ever solve that exact problem again."

Here, Taylor is suggesting what we understand to be a possible contradiction to students–that homework is meant to induce thinking and support conceptual development but, in analysis, the homework appears to be highly routine with "78 almost identical problems," in contrast to the theorems of lecture which have unique proof-structures.

Like Taylor, Payton suggested that homework is key to developing conceptual understanding, explaining, "I think, I think that's very useful. Very often, after the lecture, there are a bunch of exercises, right? So certainly, I want to ask them to practice the definition to recognize the concept, come up, maybe, with other examples, or prove your results using the definition."

Yardley also emphasized the importance of homework. Yardley was very clear that understanding meant being able to operationalize the definition of a given concept into a proof and provided a template ("elaborate scaffold") with different key parts of an epsilon-delta proof for students. Yardley noted that the way to develop this understanding is by doing the homework, explaining:

> "That's what the homework's for. That's what I mean, I tried to provide homework problems that would allow people to piece things together, such as providing that elaborate scaffold, yeah. Okay, well, I'm going to provide a structure. And then that was question one, on the homework in question two, you did not provide a scaffold, but then expected them to generalize from or like, hoped that they would learn from that scaffold… generalized to a slightly different situation. Because if you, if you look at those…you might notice that the scaffolds are actually particular to that proof, like to that statement, and so they couldn't just apply it straight away to another statement."

We viewed this proof-template scaffolding to help students "piece things together" as providing an opportunity for students to develop a connected understanding of the content and demonstrate those connections through successful completion of the homework.

4.2.2 Revisit the notes and definitions

Five of our participants highlighted the importance of reading the notes or definitions beyond the context of homework. Noel's explanation of what it means to understand was coupled with an idiosyncratic description of the actions students should take as a means of developing their understanding:

> "First I would like them to have the discipline to actually read it and read it slowly and read it multiple times. …And then I would like them to ask… 'Is this a simple definition or a complicated definition?' 'Is this definition more simple, or more complicated than definitions that I've already seen?' Even, 'is it more simple or more complicated than definitions that I've even seen in previous classes for example?' I would like comparisons of that kind to be in there with the student's thought process as much as are the particulars of what the definition actually is."

Noel specified that students should slowly read a definition multiple times and then ask themselves comparison questions about the relative difficulty or complexity of the definition as well as what the definition is about.

Quinn also emphasized the importance of revisiting material, but instead suggested that students should return to their notes and rewrite them:

> "What I tell them to do is to go back home and rewrite their notes. Okay. So what we did in class, uh, write it again by yourself … ask yourself, do I know this? Or I don't know it. And you know, they will tell you, 'I kind of know it' and I say, 'no, kind of, no, it doesn't. It means you don't know it.' So I tell him, write it and mark things where there is a gray area. You're, maybe I went to, over it fast. Maybe you missed it, but make sure you, you cross it and you come back to office hour or before class or after class. And ask me what the heck was it about?"

In this response we note that Quinn first claimed that students should rewrite their notes. Then, Quinn suggested that students should repeatedly ask "do I know this?" and insisted that "kind of" means no, suggesting that only a strong and positive "yes" is sufficient, and that students should take additional steps to develop understanding to that level. Noel similarly focused on having students "actually read it and read it slowly and read it multiple times" to make sure students really understood what they were reading.

4.2.3 Expect to struggle while learning

Eight of our participants highlighted the importance of persistence or that struggle should be expected while learning. Oakley made a set of specific claims about the ways that students should work to develop understanding of a new concept, with a focus on the notion of "getting stuck" as a critical component of learning:

> "I would want them to, to, to go over the definitions, but not read them out. I think that's the most, I think that's the common mistake that students make to read the notes and read the solutions. You read that definition, and then you read this statement of the next theorem or lemma or whatever, and then you have to sit and try to solve it yourself. And then I tell them, okay, if you don't have time to think, only 10 min, 15 min, and then read like one line of the solution. And again, so I think they consider this a waste of time. Like if they're stuck, they say, 'I didn't do anything' that was stopped the whole day. But I think those, those times that you're stuck are actually the times that you, you kind of learn really, you know, you search in your brain, you go back, you know, you're looking at your notes so that I really think that's the right thing to do."

Oakley gave a specific sequence of actions that students should take, although we do not know how to interpret the idea that students should "go over the definitions, but not read them out." Oakley does suggest that when students encounter a theorem that they try to prove it themselves rather than reading the proof; we suggest that the idea of "not reading them out" might have been foreshadowing the claim about proof. We note that Oakley specifically described the importance of getting stuck, claiming that when a person is stuck is "the times you, you kind of learn," indicating that getting stuck and then reviewing notes serves a critical role in the learning process.

Similarly, River claimed doing problems and struggle were reflective of their route to developing understanding.

> "Projecting for myself served me reasonably well, is he's just like, just don't even look at it for a day, and then come back, and then review the notes, and then start doing problems. And, and for senior level classes, doing problems [was] somewhat tricky is because there's, there's not always like, computational like, practice like that available. And then as… if you get stuck on something, go back to your notes… I suspect that a big chunk kind of come to class, kind of passively take it in and then dive straight into the homework. And then then I would hope that when they get to something and they get stuck, that then they would go back to the notes. And and you just say, 'Okay, I vaguely remember that…'"

River extends the notion of doing homework to also include "getting stuck and going back to your class notes" as a means of further development and reinforcement of ideas. We also note that River expects that students' actual practice is to "passively take it in" during class, although this is not presented as idealized practice.

Like River and Oakley, Quinn also described the homework as critical, but combined this with River's and Oakley's notion that work and struggle are critical components in the development of understanding:

> "Here's the homework, uh, apply it here. I mean, some questions in the homework are just mimic what I've done in the class. Some matters are like, I try to give them some harder question on the homework to stretch them so that they can struggle with a problem, but I help them if they struggle. I said, you know, a struggle is important in math, but I don't want infinite struggle, like work, focus on it half an hour, an hour."

Like Taylor, Quinn noted that some questions might just require mimicking or completing routine problems, but like Oakley and River, focused on the importance of struggle in the learning process.

4.3 Why activities support developing understanding

We first note that the participants did not generally specify a hypothesized mechanism for *how* mathematical activities would develop any aspects of the desired understandings. Even when asked why these activities were helpful, they typically resorted to two types of rationales, both exhibited by Skylar:

> "Well, first, I'll confess that I probably, a lot of my teaching at a baseline comes from all of the things that have always been done in the math courses that I've taken. As you well know, in math, we're rooted in tradition, and we like to do things that have worked for us. And, and we, many of us, at least continue to do the things the way that that we were brought up doing them."

Similarly, after stating that students should read the definition multiple times, Noel made two claims that help to justify this set of actions. First, "you have no idea how the student's mind works, but you do know how your own mind works." Then, because of this, Noel relies on their own learning experience:

> "If something has worked for you, then you kind of have to… I mean, you can do your best to expose how you genuinely think to the student and hope that they'll find it as useful as you found it yourself. I think that's kind of how I am."

Noel's experience as a learner further suggested that "when I'm learning stuff, I have to prepare myself psychologically or even emotionally before I can actually have the discipline to sit down and read a complicated definition… knowing that it's complicated helps me structure my study habits throughout the day." This is reflected in the repeated reading of the definition that Noel would like students to do, noted above.

Finally, we note how Taylor claims to have come to hold beliefs about learning:

> "I am influenced by the way I learned mathematics as are we all I think. But, you know, it was always the problems that taught me what's going on. I couldn't learn without, I mean, weirdly, I couldn't learn without seeing what the, what the professor had written… on the board and writing it in my notes, ideally, changing all the notation. [I] was such a stickler for notation, I couldn't bear it if their notation was objectively stupid. But you know, but I couldn't do it without that. But…the time when I was actually learning it was doing the problem sheets that we got given."

In short, Taylor claims to hold these beliefs about teaching, learning, and the importance of "thinking about the ideas" due to their experience as a mathematics learner. We interpret Taylor's claim that "it was always the problems that taught me what's going on" as a rationale for the claimed importance of homework, including the notion of 'practice' and time and effort toward a task. Fundamentally, we can understand Taylor as projecting onto the students the ways of thinking and learning that they found valuable.

Finally, Noel recognized that there is a different possible response, beyond their own experience, to why these activities are useful for students, claiming:

> "I could pretend to say, 'Okay, well, I've taught 10,000 students over the years and 749 of them genuinely learned a concept when I did this, however only 319 genuinely learned the concept when I did that.' That level of quantification I hope it strikes you as absurd."

Noel's claim might be read as indicating that numerical verification would be a more robust way of validating the utility of a particular kind of activity, but then Noel describes it as "absurd" to think that an instructor might have done this kind of work.

5 Summary and discussion

There are three broad claims that we advance about our participants. First, the participants were thoughtful about what it means to understand a concept, and highlighted the importance of both the concept definition and concept image in understanding. Second, the main activities and supports that participants suggested were repetition-focused and included a general desire for students to persist through struggle as part of learning. Third, their beliefs in the efficacy of these activities were primarily drawn from their experiences as learners and users of mathematics.

In this paper, mathematicians characterized understanding a concept both in terms of having access to a valid definition (concept definition) and in terms of a web of interconnected ideas related to the concept (concept image). With respect to the concept definition, all of our participants agreed that it was important for students to have access to a valid definition, but they varied on whether that definition should be memorized (i.e., students should memorize the formal concept definition) or re-created by students (i.e., students should create a valid personal concept definition). With respect to the concept image, participants expected understanding of a concept (the concept image) to include examples and non-examples of the concept, as well as mechanisms for knowing when to use the concept to solve problems. This included notions like being "light on one's feet" in order to apply the best tool to a problem, understanding the scope of a concept (i.e., multiple examples), and recognizing what you do and do not know. Some expressly positioned understanding as multi-faceted and expressible in multiple ways, similar to existing frameworks taught to pre-service teachers (e.g., Bloom's taxonomy; Krathwohl, 2002). This included positioning mechanical manipulation and mimicking at one end of understanding and speaking and writing coherently at the other. Generally, we note that most of the participants described rich and connected ways of thinking about a concept, including using examples to conceptualize nuance, and being able to use the concept in writing proofs. The idea that mathematicians have nuanced ways of thinking about teaching and learning is well represented in the literature [see Melhuish et al. (2022) for a review], but the notion that mathematicians have different ways of conceptualizing what coherent understanding ideally entails for any given concept is new. Moreover, while these descriptions of understanding frequently referenced knowing examples and being able to productively use the definition in writing proofs, almost all of them also included conditions such as "light on one's feet," "can speak with meaning," or "connects with things you already know" that have only a very personal definition for the mathematician.

Early writing in mathematics education (c.f., Davis and Hersh, 1981; Dreyfus, 1991) lamented that mathematicians did not have a goal of supporting students' development of conceptual understanding. Subsequent exploration, including this study, refutes those claims. The participants have a characterization of "understanding a concept" and desire that students develop conceptual understanding, but they are each slightly different and include phrases that can only have a personal meaning. One aspect of conceptual understanding shared across our participants, that students who understand a concept can re-create the definition, was often coupled with a desire that students do not memorize the definition. This aspect is both mirrored in the mathematics education literature (c.f., Vinner, 1991) and challenging. The research literature is clear that students struggle with quantification (c.f., Piatek-Jimenez, 2010), especially multiply-quantified statements (c.f., Dawkins and Roh, 2020; Vroom, 2022), suggesting that the only real way for students to be able to state the definition is to memorize it. One implication for mathematics education research is that it complicates interpreting results of large-scale studies of mathematicians' pedagogical beliefs, such as Johnson et al. (2018). Johnson et al. (2018) reported that mathematicians who teach abstract algebra wanted to promote conceptual understanding, but the researchers treated conceptual understanding as an unproblematic and unified concept. The current study suggests that significant work is needed to explore exactly what mathematicians mean by "conceptual understanding" in the context of specific courses, although this study suggests that mathematicians may not be able to fully articulate their meanings. This in some ways mirrors the development of the teacher beliefs and instructional practice literature, wherein researchers eventually called for care in mutually establishing meanings of terms like "problem solving" and "cooperative learning" between researchers and instructors (e.g., Speer, 2005). Moreover, if instructors have different meanings for understanding, as students' progress from class to class, they need to not only learn how to think in a new field of mathematics but also how to develop this new way of thinking in a manner aligned with the instructor's way of conceptualizing understanding. This suggests the need for instructors to reflect on their characterizations of understanding and then make these characterizations explicit to students so students have clarity about what types of understanding are important to the course and the teacher in particular. The act of articulating these characterizations for students may also aid instructors in reflecting on their own beliefs about understanding so that they can make additional connections between the types of understanding they value and the types of activities they encourage students to engage in.

Because each participant had a slightly different meaning for conceptual understanding, and because they have different histories and strengths as learners, they provided different activities for developing understanding. These activities included reexamining notes while questioning understanding, restating the definition with different emphases, and doing the homework. To an extent, these activities and approaches might be viewed as good advice applicable across most learning contexts. However, while lists of activities included doing the homework, prior work (Rupnow et al., 2021) argued that mathematicians crafted homework with different learning goals, and different mathematicians believed that students would learn different things from doing the same problem. Thus, while instructors shared a common "activity," even if the assigned problems were the same, the participants would be unlikely to hold the same learning goals for the particular homework assignment. When exploring the other activities that the participants suggested would be useful in developing understanding, we are left with significant questions of how the task will help develop conceptual understanding or even whether the students could know if they are executing the task with fidelity. Moreover, while Wu (1999) argued that students need to do work outside of class and do not, and Krupnik et al. (2018) and Lew et al. (2016) suggested that students do not necessarily understand the types of work and thinking needed, we argue that the situation is more complex. Specifically, each instructor's class might emphasize different aspects of understanding and associated outside-of-class activities to develop them. This raises questions about how faculty convey the nuances of their expectations about understanding and outside-of-class work to students as well as how the students apprehend those meta-mathematical ideas. This variation in instructors' goals suggests that instructors could attempt to help their students better understand their intended purpose(s) of assigned exercises and connect them to their characterizations of understanding through explicit discussion with their students, either during their lectures or when providing feedback on student work.

We also note that these activities require general approaches of repetition and persistence through struggle and that participants appeared to share the belief that work and struggle are critical in developing understanding, but they did not provide additional detail as to how. At the same time, when reading mathematics education literature (c.f., Weber and Fukawa-Connelly, 2023), the field might describe activities and intended student "markers" or "outcomes" without specifying how the activities work beyond "induce student thinking." That is, in published work, both mathematicians and mathematics educators accept the notion of 'productive struggle' as a reasonable mechanism for developing understanding, and, from a theoretical perspective, it is challenging to develop anything more specific. From the perspective of a student, how are they to judge whether they have struggled enough or in productive ways? To address this, mathematicians might reflect on and provide their students with concrete examples of instances where they struggled with a problem, ultimately found that struggle to be productive in some way, and try to articulate reasons that struggle was productive so that students may be able to extract properties of the struggle that are relevant to their work.

Moreover, because the participants held different intended learning goals, there are implications for mathematics educators. Johnson et al. (2018) and Melhuish et al. (2022) both argued that mathematics educators should attempt to design, test, and disseminate activities that accomplish the goals that teaching mathematicians, hold, but to do so implies a reasonably unified meaning for conceptual understanding and pedagogical goals. That is, this goal of developing useful and shareable interventions for mathematics education research may be unattainable. Essentially, some of the mathematics education literature (c.f., Dawkins and Weber, 2023) has recently treated mathematicians' thinking about student conceptual understanding as something mathematics educators understand and as an area where our work could be helpful. Both of these may be unwarranted beliefs of the mathematics education research community, and thus, the challenge of doing work that teaching mathematicians find useful, as a group, may be impossible. Additional research is needed to understand whether general tool development is something that can be accomplished.

Finally, the primary rationale for the mathematicians' activities and supports was, essentially, "it was useful for me." In keeping with Schön (1983), we acknowledge that the answer of 'my experience' can be sufficiently compelling for practitioners, whereas the research community may be driven to find more theory-laden answers. Moreover, Noel's comment about the absurdity of quantifying student outcomes from their courses highlights the data collection necessary for them to support claims of efficacy may be prohibitive. We also note that the authors and others have previously published numerous papers showcasing mathematicians' thoughtful, insightful understandings of mathematics and beliefs supporting their teaching (e.g., Woods and Weber, 2020; Rupnow, 2023; Rupnow and Randazzo, 2023). We thus entered this project with the assumption that mathematicians would have richly developed rationales undergirding their beliefs about understanding concepts and definitions and approaches to developing those understandings. However, our data only supports the first assumption–mathematicians provided thoughtful ways of characterizing understanding, but their explanations of how one should obtain these understandings were quite general (e.g., persist) and/or contradictory (e.g., read the definition multiple times, do not read the definition), and their views of conceptual understanding were personal enough that there does not appear to be a unified view of what conceptual understanding entails. Further research is needed to examine the affordances of different types of advice paired with different characterizations of understanding; for instance, in what contexts might reading the definition multiple times prove advantageous and for what types of activity?

Thus, we recommend researchers to carefully examine the local relationship between activities a mathematician currently uses to support students' understanding and how those activities impact their students' understandings. Future research might evaluate for whom and for developing what types of desired understandings the various described practices are useful as well as whether these views change over mathematicians' teaching careers. If some practices are more useful to develop certain ways of understanding, being explicit with students about these differences might aid them in being purposeful about their learning. Moreover, future work might attempt to develop models that explain the potentially varied utilities that the practices have.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Ethics statement

The studies involving humans were approved by Northern Illinois University Institutional Review Board. The studies were conducted in accordance with the local legislation and institutional requirements. The participants provided their written informed consent to participate in this study.

Author contributions

RR: Conceptualization, Formal Analysis, Methodology, Writing – original draft, Writing – review & editing. TF-C: Conceptualization, Formal Analysis, Methodology, Writing – original draft, Writing – review & editing.

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