Appendix A – Response statistics

For convenience of the reader, the equation of motion for the linearized system consisting of the MDOF shear-type buildings connected by linear viscous dampers equivalent to the nonlinear dampers is reported again as follows:

$$\mathbf{M}\ddot{\mathbf{U}}(t)+\left(\mathbf{C}+\mathbf{C}_d^{eq}\right)\dot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t)=-\mathbf{M}\mathbf{R}\ddot{u}_g(t) \tag{A1}$$

The Kanai-Tajimi process describing the seismic input $\ddot{u}_g(t)$ can be obtained by taking a white noise processes $w(t)$ with PSD equal to S_0 and passing it through a filter with frequency ω_{KT} and damping factor ξ_{KT}. The following equations can be used to evaluate the ground acceleration process:

$$\begin{aligned}\ddot{u}_g(t)&=2\cdot\xi_{KT}\cdot\omega_{KT}\cdot\dot{u}_{KT}(t)+\omega_{KT}^2\cdot u_{KT}(t)\\&\quad-2\cdot\xi_f\cdot\omega_f\cdot\dot{u}_g(t)-\omega_f^2\cdot u_g(t)\\\ddot{u}_{KT}(t)&+2\cdot\xi_{KT}\cdot\omega_{KT}\cdot\dot{u}_{KT}(t)+\omega_{KT}^2\cdot u_{KT}(t)=-w(t)\end{aligned} \tag{A2}$$

In order to derive the statistics of the response of the system to the stochastic input model considered, it is convenient to introduce the following state vector:

$$\mathbf{z}=\begin{bmatrix}\mathbf{u}^T & u_g & u_{KT} & \dot{\mathbf{u}}^T & \dot{u}_g & \dot{u}_{KT}\end{bmatrix}^T \tag{A3}$$

and the equation of motion of the coupled system is expressed in the first-order state space form:

$$\dot{\mathbf{z}}(t)=\mathbf{A}_s(t)\cdot\mathbf{z}(t)+\mathbf{B}_w\cdot w(t) \tag{A4}$$

where:

$$\mathbf{A}_s(t)=\begin{bmatrix}\mathbf{0}_{(n+m)x(n+m)} & \mathbf{0}_{(n+m)x1} & \mathbf{0}_{(n+m)x1} & \mathbf{I}_{(n+m)x(n+m)} & \mathbf{0}_{(n+m)x1} & \mathbf{0}_{(n+m)x1}\\ \mathbf{0}_{(n+m)x1}^T & 0 & 0 & \mathbf{0}_{(n+m)x1}^T & 1 & 0\\ \mathbf{0}_{(n+m)x1}^T & 0 & 0 & \mathbf{0}_{(n+m)x1}^T & 0 & 1\\ -\mathbf{M}^{-1}\cdot\mathbf{K} & \mathbf{R}\cdot\omega_f^2 & -\mathbf{R}\cdot\omega_{KT}^2 & -\mathbf{M}^{-1}\cdot\left(\mathbf{C}+\mathbf{C}_{ds}^{eq}(t)\right) & \mathbf{R}\cdot 2\cdot\xi_f\cdot\omega_f & -\mathbf{R}\cdot 2\cdot\xi_{KT}\cdot\omega_{KT}\\ \mathbf{0}_{(n+m)x1}^T & -\omega_f^2 & \omega_{KT}^2 & \mathbf{0}_{(n+m)x1}^T & -2\cdot\xi_f\cdot\omega_f & 2\cdot\xi_{KT}\cdot\omega_{KT}\\ \mathbf{0}_{(n+m)x1}^T & 0 & -\omega_{KT}^2 & \mathbf{0}_{(n+m)x1}^T & 0 & -2\cdot\xi_{KT}\cdot\omega_{KT}\end{bmatrix} \tag{A5}$$

where $\mathbf{0}_{(n+m)x(n+m)}$ is a $(n+m)$ x $(n+m)$ matrix of zeros, $\mathbf{0}_{(n+m)x1}$ = $(n+m)$ x 1 vector of zeros, and

$$\mathbf{B}_w(t)=\begin{bmatrix}\mathbf{0}_{(n+m)x1}^T & 0 & 0 & \mathbf{0}_{(n+m)x1}^T & 0 & -1\end{bmatrix}^T \tag{A6}$$

The stochastic response can be obtained by solving the differential equation of the covariance matrix $\mathbf{Q_{ZZ}}(t)$, whose time-varying elements are the second order moments $E\left[Z_i(t)\cdot Z_j(t)\right]$ relative to the state vector $\mathbf{Z}$:

$$\dot{\mathbf{Q}}_{\mathbf{zz}}(t)=\mathbf{A}_s(t)\cdot\mathbf{Q}_{\mathbf{zz}}(t)+\mathbf{Q}_{\mathbf{zz}}(t)\cdot\mathbf{A}_s(t)^T+\mathbf{G}(t) \tag{A7}$$

Here, matrix $\mathbf{G}(t)$ has all elements equal to zero, except for $\mathbf{G}_{2\cdot(n+m)+2,2\cdot(n+m)+2}(t)=2\cdot\pi\cdot S_0$. Matrix $\mathbf{Q}_{\mathbf{zz}}(t)$ contains information about the statistics of the displacement and velocity response.

Since the stationary response is of interest, $\dot{\mathbf{Q}}_{\mathbf{ZZ}}(t)$=0, and Eqn.(A7) reduces to: $-\mathbf{M}^{-1}\cdot\left(\mathbf{C}+\mathbf{C}_{ds}^{eq}(t)\right)$

$$\mathbf{0}=\mathbf{A}_s\mathbf{Q}_{\mathbf{ZZ}}(t)+\mathbf{Q}_{\mathbf{ZZ}}(t)\mathbf{A}_s{}^T+\mathbf{G}(t) \tag{A8}$$

It is noteworthy that the term of matrix $\mathbf{A}_s(t)$ is function of the response covariance, and thus an iterative scheme is required to solve Eq. (A8). Obviously, in the case of linear dampers, no iterations are required, and the problem solution could alternatively be found in the frequency domain (see e.g. Gioiella et al. 2018).
Matrix $\mathbf{Q_{ZZ}}$ contains the statistics of the displacement and velocity response at the various floors. The equivalence between the linear and nonlinear FVDs is based on the knowledge of the standard deviation of the relative velocities at the various floors, $\sigma_{\Delta\dot{\mathbf{U}}_j}$, for j=1,2,..,n. To determine these, the linear operator $\mathbf{D}_{\Delta\dot{\mathbf{U}}}$ is introduced to relate the relative velocities $\Delta\dot{\mathbf{U}}$ to $\mathbf{Z}$ through the expression $\Delta\dot{\mathbf{U}}=\mathbf{D}_{\Delta\dot{\mathbf{U}}}\mathbf{Z}$. The values of $\sigma_{\Delta\dot{\mathbf{U}}_j}$ can be found along the diagonal of the following covariance matrix:

$$\mathbf{Q}_{\Delta\dot{\mathbf{U}}\Delta\dot{\mathbf{U}}}(t)=\mathbf{D}_{\Delta\dot{\mathbf{U}}}\cdot\mathbf{Q}_{\mathbf{zz}}(t)\cdot\mathbf{D}_{\Delta\dot{\mathbf{U}}}^T \tag{A9}$$

The covariance matrix of the absolute accelerations of the system, collected in the vector $\mathbf{a}(t)$, is obtained as:

$$\mathbf{Q}_{\mathbf{aa}}(t)=\mathbf{D}_{\mathbf{a}}(t)\cdot\mathbf{Q}_{\mathbf{zz}}(t)\cdot\mathbf{D}_{\mathbf{a}}^T(t) \tag{A10}$$

where matrix $\mathbf{D}_{\mathbf{a}}(t)$, defined as:

$$\mathbf{D}_{\mathbf{A}}(t)=-\mathbf{M}^{-1}\cdot\left[\mathbf{K}\quad\mathbf{0}_{(n+m)x1}\quad\mathbf{0}_{(n+m)x1}\quad\mathbf{C}+\mathbf{C}_{ds}^{eq}(t)\quad\mathbf{0}_{(n+m)x1}\quad\mathbf{0}_{(n+m)x1}\right] \tag{A11}$$

is such that $\mathbf{A}(t)=\ddot{\mathbf{U}}(t)+\mathbf{r}\cdot\ddot{U}_g(t)=\mathbf{D}_{\mathbf{A}}(t)\cdot\mathbf{Z}(t)$.

The covariance matrix of the damper forces, collected in the vector $\mathbf{F}_d(t)$, is obtained as:

$$\mathbf{Q}_{\mathbf{FF}}(t)=\mathbf{D}_{\mathbf{F}}(t)\cdot\mathbf{Q}_{\mathbf{zz}}(t)\cdot\mathbf{D}_{\mathbf{F}}{}^T(t) \tag{A12}$$

where matrix $\mathbf{D}_{\mathbf{F}}(t)$ is defined as:

$$\mathbf{D_F}(t) = \begin{bmatrix} \mathbf{0}_{(n+m)x(n+m)} & \mathbf{0}_{(n+m)x1} & \mathbf{0}_{(n+m)x1} & \mathbf{C}_{ds}^{eq}(t) & \mathbf{0}_{(n+m)x1} & \mathbf{0}_{(n+m)x1} \end{bmatrix} \tag{A13}$$

Appendix B – Validation of linearization approach via Monte Carlo simulation

The accuracy of the results obtained through the proposed analysis technique is evaluated based on the comparison with Monte Carlo Simulation (MCS) results. The MCS estimates are obtained by considering a set of 200 ground motion records compatible with the with the Kanai-Tajimi input model (Kanai 1957, Tajimi 1960) and generated by using the Spectral representation method (Shinozuka and Deodatis 1991).
The case of buildings connected by a single nonlinear damper (α = 0.3 and optimal properties as reported in Table 2) at the 4th floor is considered, and the relevant results are shown in Figure B1, where the stationary response evaluated by solving the Lyapunov equation and applying the stochastic linearization technique is compared to the stationary response evaluated via MCS.
Comparison is made in terms of mean square response of the displacement of building B at floor 4 (Figure B1a), and of the damper force (Figure B1b).
It is observed that the stochastic linearization technique provides very accurate response estimates despite the high level of nonlinearity of the dampers.

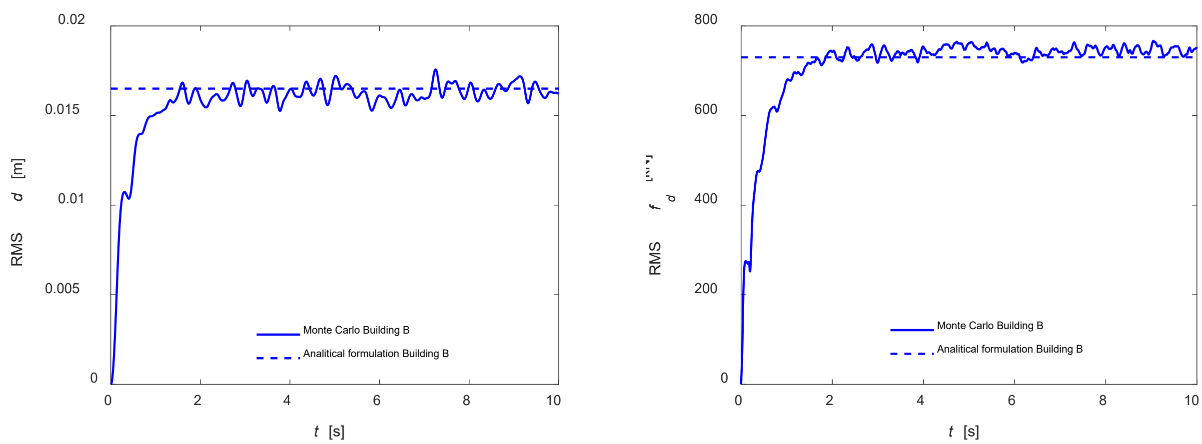

Figure B1. Comparison of stationary mean square response according to analytical formulation and MCS in terms of a) displacements of building A and B at floor 4, b) damper force at floor 4. Case with one single damper at the 4th floor with α = 0.3 and optimal properties.

References

Shinozuka, M., and Deodatis, G. (1991). Simulation of stochastic processes by spectral representation. *Applied Mechanics Review* 1991; 44, 191-203, doi: https://doi.org/10.1115/1.3119501.