Appendix A: Explicit derivation of the Propagator of Delta Potentials for the Case: λ is constant

In this part we will show that if the strength of δ potential given in Eq. (13) is constant $\lambda(t)=\lambda$, it is possible to obtain an explicit expression for the Green's function or the propagator and therefore to get an explicit expression for the wave function for all times $t>0$. Since λ is constant, $\mathcal{L}\left\{\lambda(t)\psi(0,t)\right\}=\lambda\mathcal{L}\left\{\psi(0,t)\right\}=\lambda\bar{\psi}(0,s)$. Using this, we write Eq. (17) as

$$\bar{\psi}(x,s)=\frac{1}{2\sqrt{is}}\int_{-\infty}^{\infty}dx'\ e^{i\sqrt{is}|x-x'|}\psi(x',0)-\frac{i\ \lambda}{2\sqrt{is}}e^{i\sqrt{is}|x|}\bar{\psi}(0,s)\ . \tag{A.1}$$

Writing this equation for $x=0$ and solving it for $\bar{\psi}(0,s)$, we get

$$\bar{\psi}(0,s)=\frac{1}{2\sqrt{is}+i\lambda}\int_{-\infty}^{\infty}dx'\ e^{i\sqrt{is}|x'|}\psi(x',0)\ . \tag{A.2}$$

After inserting this expression back into the Eq.(A.1) we find

$$\bar{\psi}(x,s)=\int_{-\infty}^{\infty}dx'\ \left(\frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}}-\frac{\lambda e^{i\sqrt{is}(|x|+|x'|)}}{2\sqrt{s}(2\sqrt{s}+\sqrt{i}\lambda)}\right)\psi(x',0)\ . \tag{A.3}$$

The factor multiplying $\psi(x',0)$ in the integral is Laplace transform of the Green's function in time variable:

$$\bar{G}(x,x',s)=\left(\frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}}-\frac{\lambda e^{i\sqrt{is}(|x|+|x'|)}}{2\sqrt{s}(2\sqrt{s}+\sqrt{i}\lambda)}\right)\ . \tag{A.4}$$

By taking the inverse Laplace transform of this expression we get

$$G(x,x',t)=\frac{1}{\sqrt{4\pi it}}\exp\left[\frac{i(x-x')^2}{4t}\right]-\frac{\lambda}{4}\exp\left[\frac{\lambda}{2}(|x|+|x'|)+i\frac{\lambda^2}{4}t\right]\mathrm{erfc}\left[\frac{(|x|+|x'|)}{2\sqrt{it}}+\sqrt{it}\frac{\lambda}{2}\right]\ , \tag{A.5}$$

where we have used the integral representation $\mathrm{erfc}[-b/(2\sqrt{a})]=e^{-b^2/(4a)}\sqrt{a/\pi}\ \int_0^{\infty}dx\,e^{-ax^2+bx}$ and $\mathrm{erfc}(x)=1-\mathrm{erf}(x)=(2/\sqrt{\pi})\int_x^{\infty}dx'\ e^{-x'^2}$. Note that the initial value problem is expressed at time $t=0$, but the wave function can be given at any initial time $t=t_0$ and all the expressions in t is replaced by $t-t_0$. Another common notation for Green's function is $G(x,t;x',t_0)$. Hence, we obtain

$$\begin{aligned}\psi(x,t)=\int_{-\infty}^{\infty}dx'\ \Bigg\{&\frac{1}{\sqrt{4\pi it}}\exp\left[\frac{i(x-x')^2}{4t}\right]\\ &-\frac{\lambda}{4}\exp\left[\frac{\lambda}{2}(|x|+|x'|)+i\frac{\lambda^2}{4}t\right]\mathrm{erfc}\left[\frac{(|x|+|x'|)}{2\sqrt{it}}+\sqrt{it}\ \frac{\lambda}{2}\right]\Bigg\}\psi(x',0)\ .\end{aligned} \tag{A.6}$$

This result is consistent with the ones obtained by different methods [24,25,36].

Appendix B: Explicit derivation of the Propagator of Delta Potentials for the Case: $\lambda(t)=\alpha/t$

We review the explicit propagator derivation of Dirac delta potential for strengths with inversely proportional in time originally given in [36]. The general expression (9) for the Laplace transform of the wave function in this particular case $\lambda(t)=\frac{\alpha}{t}$, where α is a constant, becomes

$$\bar{\psi}(x,s)=\frac{1}{2\sqrt{is}}\int_{-\infty}^{\infty}dx'\ e^{i\sqrt{is}|x-x'|}\psi(x',0)-\frac{i}{2\sqrt{is}}e^{i\sqrt{is}|x|}\ \mathcal{L}\left\{\frac{\alpha}{t}\psi(0,t)\right\}\ . \tag{B.1}$$

Note that

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty ds' \ \mathcal{L}\{f(t)\} \ . \tag{B.2}$$

Using this result and choosing $x=0$ in Eq. (B.1) we get

$$\bar{\psi}(0,s) = \frac{\alpha}{2i\sqrt{is}}\int_s^\infty ds' \ \bar{\psi}(0,s') + \frac{1}{2\sqrt{is}}\int_{-\infty}^\infty dx' \ e^{i\sqrt{is}|x'|}\psi(x',0) \ . \tag{B.3}$$

If we define

$$u(s) = \int_s^\infty ds' \ \bar{\psi}(0,s') \ , \tag{B.4}$$

the Eq.(B.1) turns out to be

$$\bar{\psi}(0,s) = \frac{\alpha}{2i\sqrt{is}}u(s) + \frac{1}{2\sqrt{is}}\int_{-\infty}^\infty dx' \ e^{i\sqrt{is}|x'|}\psi(x',0) \ . \tag{B.5}$$

Since $\bar{\psi}(0,s) = -\frac{du(s)}{ds}$, the above expression yields

$$\frac{du(s)}{ds} + \frac{\alpha}{2i\sqrt{is}}u(s) = -\frac{1}{2\sqrt{is}}\int_{-\infty}^\infty dx' \ e^{i\sqrt{is}|x'|} \ \psi(x',0) \ . \tag{B.6}$$

The solution of this first order differential equation gives

$$u(s) = \int_{-\infty}^\infty dx' \ \left(\frac{e^{i\sqrt{is}|x'|}}{i\alpha + |x'|}\right) \ \psi(x',0) \ . \tag{B.7}$$

By taking the derivative under the integral sign, we get

$$\bar{\psi}(0,s) = \frac{1}{2\sqrt{is}}\int_{-\infty}^\infty dx' \ \left(\frac{|x'| \ e^{i\sqrt{is}|x'|}}{|x'| + i\alpha}\right) \ \psi(x',0) \ . \tag{B.8}$$

Using the Eq. (10), we can write

$$\psi(0,t) = \frac{1}{2\sqrt{i\pi t}}\int_{-\infty}^\infty dx' \ \left(\frac{|x'|e^{i\frac{x'^2}{4t}}}{|x'| + i\alpha}\right) \ \psi(x',0) \ . \tag{B.9}$$

Inserting this expression for $\psi(0,t)$ in the Laplace transform of $\frac{\alpha}{t}\psi(0,t)$ and evaluating the t integral, we get

$$\mathcal{L}\left\{\frac{\alpha}{t}\psi(0,t)\right\} = \int_{-\infty}^\infty dx' \ \left(\frac{\alpha \ e^{i\sqrt{is}|x'|}}{|x'| + i\alpha}\right) \ \psi(x',0) \ . \tag{B.10}$$

We substitute this result into Eq.(B.1) and obtain $\bar{\psi}(x,s)$ as

$$\bar{\psi}(x,s) = \int_{-\infty}^\infty dx' \ \frac{1}{2\sqrt{is}}\left[e^{i\sqrt{is}|x-x'|} - i\alpha \ \left(\frac{e^{i\sqrt{is}(|x|+|x'|)}}{|x'| + i\alpha}\right)\right] \ \psi(x',0) \ . \tag{B.11}$$

Thus the Green' s function written in terms of the s variable is

$$\bar{G}(x,x',s) = \frac{1}{2\sqrt{is}}\left[e^{i\sqrt{is}|x-x'|} - i\alpha \ \left(\frac{e^{i\sqrt{is}(|x|+|x'|)}}{|x'| + i\alpha}\right)\right] \ . \tag{B.12}$$

This function is easily transformed back using Eq. (10) to get the propagator

$$G(x,x',t) = \frac{1}{\sqrt{4\pi i t}}\left[\exp\left[\frac{i(x-x')^2}{4t}\right] - \frac{i\alpha}{|x'| + i\alpha} \ \exp\left[i\frac{(|x| + |x'|)'^2}{4t}\right]\right] \ . \tag{B.13}$$

Therefore the wave function for this case is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \, \left\{ \frac{1}{\sqrt{4\pi i t}} \left[\exp\left[\frac{i(x-x')^2}{4t} \right] - \frac{i\alpha}{|x'| + i\alpha} \, \exp\left[i \frac{(|x| + |x'|)^2}{4t} \right] \right] \right\} \psi(x', 0) \; . \tag{B.14}$$

One can also find a closed analytic expression of the propagator for exponentially decaying strengths written in terms of an infinite product [36].