1. **Appendix 1: Landau-de Gennes Free Energy. Governing Equations: derivations**

The purpose of this appendix is to derive Eq. (2.7). The scalar form of the Eq. (2.6) is presented below:



The following notationis adopted for partial derivatives of a given function with respect to a given *k*. So, for , it is presented a list up to the third derivative of *F* with respect to the order parameters .



The minimization of the free energy in Eq. with respect to each order parameter yields a system of differential equations, which given the conditions in Eq. , provides the different phases considered. Solving for q at those conditions and plugging it back to Eq. , a reparametrized LdG for the I-SmA is obtained:



The numerical values of the coefficients are presented in appendix A.4.

From this definition, a list of the first, second and third derivatives is provided:



1. **Differential Geometry**

This appendix section expands on the tensorial definition for the differential geometry to obtain the equations in Section 2.2. The derivatives are taken from appendix A.1.

The Monge parametrization is used to obtain the surface tangent vectors  which comes from the position vector on the energy landscape :





The surface metric tensor is described as the first fundamental form as:



which also induces the arc-length ds:



with a determinant , Thus, the metric tensor is invertible so that:



For convenience, the elements of the surface metric are renamed with the notation of the first and second fundamental coefficients:





Using the notation from Eqs. - and considering that the determinant of the surface metric is invertible, the second fundamental form, known as the symmetric curvature  tensor, is:



Here the  tensor bases defined by the identity tensor **I** and the traceless surface deviatoric tensor **q** define the mean (*H*), and deviatoric (*D*) curvatures considering no off-diagonal () and tensor alternator () contributions gives  (Wang et al., 2020). Then, from the mixed form  of , the principal curvatures, mean (*H*) and gaussian (*K*) respectively, are:



where a given value for *D* characterizes the non-sphericity (). Finally, the deviatoric curvature (*D*), Casorati (*C*), and shape coefficient (*S*) are:



1. **Computational Methods for Geodesics**

This appendix section we include the algorithm used to compute the geodesics equations .

The equation for the geodesic lines are written in compact form (where  represents the arc-length derivative), and the first fundamental form induces the arc-length :



where , are the Christoffel symbols defined by:



Or in the component form for the order parameter frame:



The arc length is computed using:



where the coefficients *E*, *A*, and *G* are from .

1. **Landau-de Gennes coefficient values**

This appendix presents a list of the coefficient values used throughout this paper and first introduced in Figure 2 for 12CB (Abukhdeir and Rey, 2008, Coles and Strazielle, 2011, Urban et al., 2005):

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