

# Supplementary material for ‘Transport of melt, pressure and heat through a magma mush’

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# 1 Governing equations and solution method

Table 1: Symbols of the constants used in the study

	symbol	definition	typical value/expression
constants	$P_{abs}$	absolute pressure of the system	100MPa
	$\alpha$	poro-elastic coefficient (Biot coefficient) of the mush	0.6, 0.9
	$\phi_o$	porosity in the mush at initial steady state	0.3
	$\chi$	pore gas volume fraction	0-0.3
	$K_s$	elastic bulk modulus of solid crystals	10GPa
	$K_l$	bulk modulus of pure magma	1Gpa
	$K_g$	bulk modulus of gas	$P_{abs}$
	$K_{pore}$	bulk modulus of pore magma	$1/(\chi/K_g + (1-\chi)/K_l)$
	$K_m$	drained bulk modulus of the mush	$(1-\alpha)K_s$
	$K_u$	undrained bulk modulus of the mush	$(1-\alpha)K_s + \frac{\alpha^2 K_s K_{pore}}{\phi_o K_s + (\alpha - \phi_o) K_{pore}}$
	$S$	poroelastic storage coefficient	$\frac{K_u + \frac{4}{3}\mu}{K_m + \frac{4}{3}\mu} \frac{\alpha^2}{K_u - K_m}$
	$\gamma$	factor governing poroelastic stress-pressure coupling	$\frac{K_u - K_m}{\alpha(K_u + \frac{4}{3}\mu)}$
	$A$	constant	$\frac{K_u}{K_u + \frac{4}{3}\mu}$
	$B$	constant	$\frac{K_m}{K_m + \frac{4}{3}\mu}$
	$\beta_m$	volumetric thermal expansion coefficient of pure magma	$5 \times 10^{-5} / ^\circ C$
	$\beta_g$	volumetric thermal expansion coefficient of gas	$1/T$
	$\beta_s$	volumetric thermal expansion coefficient of solid crystals	not explicitly used
	$\beta_{pore}$	volumetric thermal expansion coefficient of the pore	$\chi\beta_g + (1-\chi)\beta_l$
	$\eta_f$	viscosity of pore magma	not explicitly used
	$\kappa$	permeability in the mush	not explicitly used
	$\mu$	instantaneous shear modulus of the host rock and the mush	1GPa
	$\eta$	viscosity of the mush ensemble	not explicitly used
	$\tau_{relax}$	relaxation time	not explicitly used
	$[l]$	characteristic length $\sqrt{\tau_{relax} c}$	$< 10\text{Km}$
	$\rho_f$	magma density	not explicitly used
	$\rho_m$	crystalline matrix density	not explicitly used
	$c_f$	magma specific heat	not explicitly used
	$c_m$	crystalline matrix specific heat	not explicitly used
	$\delta$	dimensionless value $\delta \equiv c_f \rho_f / c_m \rho_m$	1
	$c$	poro-elastic diffusivity	not explicitly used
	$\kappa_T$	thermal diffusivity in much	not explicitly used
	$R$	dimensionless value $c/\kappa_T$	100, $\infty$
	$\lambda_P$	poroelastic skin depth $\sqrt{2c/\omega}$	not explicitly used
	$\lambda_T$	thermal skin depth $\sqrt{2\kappa_T/\omega}$	not explicitly used
$\Delta$	dimensionless value	0-2	
$De$	Deborah number $\omega\eta/\mu$	$10^{-3} - 10^3$	
$\omega$	frequency for harmonic perturbations	not explicitly used	
$D_T$	background temperature gradient	0-500 $^\circ C/\text{Km}$	
variables	$\sigma_{ij}$	stress tensor	
	$\epsilon_{ij}$	strain tensor	
	$\epsilon$	volumetric strain	
	$P$	pore pressure	
	$\zeta$	variation of fluid content	
	$\vec{q}$	Darcy's flow velocity	
	$T$	temperature deviation from background value	
	$\hat{T}(z)$	Fourier transform for temperature deviation	
	$\hat{P}(z)$	Fourier transform for pressure	
$\hat{q}(z)$	Fourier transform for fluid velocity		

Table 1 shows the values and variables in this study. The thermo-poro-viscoelastic constitutive relations governing the deformation of the mush and the pore magma content are similar to Liao [2022]. The strain-stress relations are [Cheng, 2016, Biot, 1941]

$$\dot{\sigma}_{ij} + \frac{\mu}{\eta}\sigma_{ij} = \frac{\mu}{\eta}(K_m\epsilon - \alpha P)I + 2\mu\dot{\epsilon}_{ij} + (K_m - \frac{2}{3}\mu)\dot{\epsilon}I - \alpha\dot{P}I - K_m\beta_s\dot{T}I \quad (1a)$$

$$\zeta = \alpha\epsilon + \frac{\alpha^2}{K_u - K_m}P - (\phi\beta_{pore} + (\alpha - \phi)\beta_s)T \quad (1b)$$

where the overhead dot  $\dot{\phantom{x}}$  denotes partial derivative in time,  $I$  denotes identity matrix.  $\sigma_{ij}$  and  $\epsilon_{ij}$  are stress and strain tensors of the ensemble material,  $\epsilon$  is the volumetric strain,  $P$  is the pore pressure,  $T$  is the temperature variation from its reference value.  $\zeta$  is the variation of fluid content, defined as the increment of pore fluid volume per un-deformed volume of the mush.  $\mu$  and  $\eta$  are the shear modulus and shear viscosity of the crystalline framework,  $\alpha$  is the Biot coefficient of poroelasticity, and  $\phi$  is the porosity in the mush.  $\beta_s$  is the volumetric thermal expansion coefficient for the solid crystals. The thermal expansion coefficient of the gas-rich pore magma  $\beta_{pore} = (1 - \chi)\beta_m + \chi\beta_g$ , where  $\beta_g = 1/T_{gas}$  is the thermal expansion coefficient of ideal gas.  $K_u$  and  $K_m$  are the undrained and drained bulk moduli.

The equilibrium condition, Darcy's law, mass conservation, and energy conservation are

$$\nabla \cdot \sigma_{ij} = 0 \quad (2a)$$

$$\vec{q} = -\frac{\kappa}{\eta_f}\nabla P \quad (2b)$$

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot \vec{q} = 0 \quad (2c)$$

$$\frac{\partial T}{\partial t} + \frac{\rho_f c_f}{\rho_m c_m}\vec{q} \cdot \nabla T - \kappa_T \nabla^2 T = 0 \quad (2d)$$

where  $\vec{q}$  is Darcy's flow velocity (assumed to only have the vertical component),  $\kappa$  is the permeability of the mush,  $\eta_f$  is magma viscosity.  $(\rho_f, c_f)$  and  $(\rho_m, c_m)$  are the density and specific heat of the fluid phase and of the whole mush ensemble, respectively. The value of  $\frac{\rho_f c_f}{\rho_m c_m}$  goes not significantly change the results and assume it to be 1 in the rest of the study.  $\kappa_T$  is the thermal diffusivity in the mush.

I consider a 1D problem where pressure and temperature perturbation at a given location (i.e., a magmatic lens) can be expressed as a smooth function in time, and observe how the perturbation propagate up and down the mush column. A zeroth-order background state of the mush column consists of no melt flows, balance between lithospheric stress and interior stress, and a linear temperature profile that has a negative or zero gradient (i.e., uniform temperature of a temperature that decreases linearly with distance from the bottom to top of the mush column). All terms in the equations (1) and (2) are linear, except for the thermal advection (second term in Equation 2d), which I linearize to first-order using the background temperature gradient. Cast the new linearized equations in frequency-domain using Fourier transform, I find that the Fourier transforms for pressure, velocity, and temperature for all the 1-st order deviations obey [Cheng, 2016, Liao, 2022]

$$i\omega\hat{P} - c^*\hat{P}'' = -\gamma^*i\omega\hat{\sigma}_{zz} + i\omega\frac{\beta_c}{S^*}\hat{T} \quad (3a)$$

$$\hat{q} = -\frac{\kappa}{\eta_f}\hat{P}' \quad (3b)$$

$$i\omega\hat{T} - D_T\hat{q} = \kappa_T\hat{T}'' \quad (3c)$$

where the Fourier transform of vertical stress  $\hat{\sigma}_{zz} = -\hat{P}_o$  is uniform in space for an uni-axial column loaded by pressure perturbation  $\hat{P}_o$  at the source ( $z = 0$ ). The 0-th order (negative) background temperature gradient is  $dT^{(0)}/dz = -D_T$  (with unit of  $^{\circ}C/m$ ). The effective thermal expansion coefficient  $\beta_c$  is constructed from volumetric expansion coefficients of both the gas-bearing fluid phase and the solid phase (see Table 1). It is worth noting that the viscoelastic effect is incorporated by correspondence principle: the Fourier transform of the governing equations are identical to their elastic counterparts with the exception of shear modulus, which is replaced by a complex rigidity  $\mu^*$  (similar in concept to s-dependent rigidity under Laplace transform )

$$\mu^* = \mu \frac{i\omega\tau_{relax}}{1 + i\omega\tau_{relax}} \quad (4)$$

where  $\tau_{relax}$  is the relaxation time. We can verify that when  $\tau_{relax} = \infty$ , the complex rigidity becomes  $\mu$ . The complex rigidity in turn leads to complex coefficients

$$\gamma^* = \frac{K_u - K_m}{\alpha(K_u + \frac{4}{3}\mu^*)} = \gamma \frac{1 + i\omega\tau_{relax}}{A + i\omega\tau_{relax}}, \quad S^* = \frac{K_u + \frac{4}{3}\mu^*}{K_m + \frac{4}{3}\mu^*} \frac{\alpha^2}{K_u - K_m} = S \frac{A + i\omega\tau_{relax}}{B + i\omega\tau_{relax}} \quad (5)$$

where

$$A \equiv \frac{K_u}{K_u + \frac{4}{3}\mu}, \quad B \equiv \frac{K_m}{K_m + \frac{4}{3}\mu}, \quad \gamma \equiv \frac{K_u - K_m}{\alpha(K_u + \frac{4}{3}\mu)}, \quad S \equiv \frac{K_u + \frac{4}{3}\mu}{K_m + \frac{4}{3}\mu} \frac{\alpha^2}{K_u - K_m} \quad (6)$$

The first two relations in (3) can further lead to evolution of fluid velocity

$$i\omega\hat{q} - c^*\hat{q}'' = -c^*\beta_c i\omega\hat{T}' \quad (7)$$

where the poroelastic diffusivity  $c$  is modified by viscoelastic relaxation to a complex value

$$c^* = \frac{\kappa}{\eta_f} \frac{(K_u - K_m)(K_m + \frac{4}{3}\mu^*)}{\alpha^2(K_u + \frac{4}{3}\mu^*)} = c \frac{\frac{K_m}{K_m + \frac{4}{3}\mu^*} + i\omega\tau}{\frac{K_u}{K_u + \frac{4}{3}\mu^*} + i\omega\tau} = c \frac{B + i\omega\tau_{relax}}{A + i\omega\tau_{relax}} \quad (8)$$

In the main text, a Deborah number  $De = \omega\tau_{relax}$ , which becomes infinitely large when there is no relaxation. We can verify that when  $De = \infty$ ,  $\gamma^*$ ,  $c^*$ ,  $S^*$  all become real and recover the definition form poroelastic theories.

## 1.1 Frequency-domain solution for a thermo-poroelastic mush with no thermal diffusion

The effect of thermal mechanical coupling and their effect on perturbation propagation can be most clearly seen in an simplified end-member case of thermal-poroelasticity with vanishing thermal diffusion and viscous relaxation. With  $\kappa_T = 0, \omega\tau_{relax} = \infty$ , (1) and (2) lead to the evolution for pressure, velocity, and temperature perturbations

$$\frac{\partial P}{\partial t} - c \frac{\partial^2 P}{\partial z^2} = \frac{\beta_c}{S} \frac{\partial T}{\partial t} - \gamma \frac{\partial \sigma_{zz}}{\partial t} \quad (9a)$$

$$q = -\frac{\kappa}{\eta_f} \frac{\partial P}{\partial z} \quad (9b)$$

$$\frac{\partial T}{\partial t} - D_T q = 0 \quad (9c)$$

with the storage coefficient  $S$  and  $\gamma$  are determined by the poroelastic properties of the mush (see Table 1). The system defined by (9) further results in a diffusion-advection equation for pressure

$$\frac{\partial P}{\partial t} - c \frac{\partial^2 P}{\partial z^2} + c\beta_c \delta D_T \frac{\partial P}{\partial z} = -\gamma \frac{\partial \sigma_{zz}}{\partial t} \quad (10)$$

The solution to (10) is the sum of the special solution  $-\gamma \sigma_{zz}(t)$  and a general solution to the homogeneous equation (with 0 on the RHS). The general solution is the superposition of waveform solutions  $e^{i\omega t + kz}$ , with wavenumber  $k$  determined by the dispersion relation

$$k^2 - \beta_c D_T k - \frac{i\omega}{c} = 0 \quad (11)$$

where  $D_T \geq 0$  is the opposite of the background temperature gradient. A purely diffusive system ( $D_T = 0$ ) for poroelastic medium results in  $k_P^2 = i\omega/c$ , which corresponds to an e-folding decay length  $\lambda_P = \sqrt{2c/\omega}$ . With arbitrary value for  $D_T$ , the solution for (11) is

$$k\lambda_P = \frac{\Delta}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + 2i} \quad (12)$$

where  $\Delta = \beta_c D_T \lambda_P$  is the dimensionless background thermal gradient normalized by the length scale  $\lambda_P$  and temperature scale  $\beta_c^{-1}$ . We can verify that when  $\Delta = 0$ , the diffusive endmember has  $k_P \lambda_P = \pm(1 + i)$ . In a 1D boundary-less medium, the two wave-numbers prescribe the propagation of a signal away from the source [Turcotte and Schubert, 2002]. The wavenumber with positive real part describes 'top-down' propagation in the lower ( $z < 0$ ) domain; the wavenumber with negative real part describes 'bottom-up' propagation from the source into the upper ( $z > 0$ ) domain. For the given source perturbation  $P(0) = P_o e^{i\omega t}$ , and a fluid-loading boundary condition such that  $\sigma_{zz}(t) = -P(0, t)$ , the amplitude and wavenumber for each domain are determined. The governing equation (10) has frequency-domain solution

$$\hat{P}(z)/P_o = (1 - \gamma) e^{\Delta z/2\lambda_P - \sqrt{(\Delta/2)^2 + 2i}|z|/\lambda_P} + \gamma \quad (13)$$

which leads to time-domain solution  $P(z, t) = \hat{P}(z) e^{i\omega t}$ . The frequency-domain solution for pressure leads to frequency-domain solutions for temperature and fluid velocity that evolve according to (9), which are

$$\hat{T}\beta_c/P_o S = \frac{\Delta}{2} i \left( \frac{\Delta}{2} - \frac{z}{|z|} \sqrt{(\Delta/2)^2 + 2i} \right) e^{\Delta z/2\lambda_P - \sqrt{(\Delta/2)^2 + 2i}|z|/\lambda_P} \quad (14a)$$

$$\frac{q}{\omega\lambda_P}/P_o S = -\frac{1}{2} \left( \frac{\Delta}{2} - \frac{z}{|z|} \sqrt{(\Delta/2)^2 + 2i} \right) e^{\Delta z/2\lambda_P - \sqrt{(\Delta/2)^2 + 2i}|z|/\lambda_P} \quad (14b)$$

where  $\beta_c^{-1}$ ,  $S^{-1}$ , and  $\omega\lambda_P$  arise naturally as scales for temperature, pressure and velocity.

## 1.2 Frequency-domain solutions with full thermo-poro-viscoelastic rheology

The frequency-domain solutions for a fully thermo-poro-viscoelastic mush column with finite viscoelastic relaxation and thermal diffusivity obey the evolution equations (3) and can be expressed as

$$\hat{P}(\omega) = \gamma^* \hat{P}_o(\omega) + \Sigma \hat{P}_k e^{kz} \quad (15a)$$

$$\hat{T}(\omega) = \Sigma_k \hat{T}_k e^{kz} \quad (15b)$$

$$\hat{q}(\omega) = \Sigma_k \hat{q}_k e^{kz} \quad (15c)$$

with boundary conditions

$$\Sigma_k \hat{P}_k(\omega, k) = \hat{P}_o(\omega)(1 - \gamma^*) \quad (16a)$$

$$\Sigma_k \hat{T}_k(\omega, k) = \hat{T}_o \quad (16b)$$

where  $\hat{T}_o$  is the Fourier amplitude of temperature perturbation at 0. Substituting the wavenumber decomposition into the evolution of momentum and energy equation

$$\begin{bmatrix} i\omega - c^* k^2 & -\frac{\beta_c}{S^*} i\omega & 0 \\ \frac{\kappa}{\eta_f} k & 0 & 1 \\ 0 & i\omega - \kappa_T k^2 & -D_T \end{bmatrix} \begin{bmatrix} \hat{P}_k \\ \hat{T}_k \\ \hat{q}_k \end{bmatrix} = 0 \quad (17)$$

### 1.2.1 Dispersion relation for propagation of harmonic signals

For a system perturbed by harmonic signals, the poroelastic skin depth  $\lambda_P$  and frequency  $\omega$  naturally arise as length and time scales, as shown in section §1.1. I retain these scales for the general case where thermal diffusion and viscolastic relaxation are present, and re-express (17) as

$$\begin{bmatrix} i - \frac{1}{2} \frac{B+iDe}{A+iDe} (k\lambda_P)^2 & -\frac{B+iDe}{A+iDe} i & 0 \\ \frac{1}{2} \Delta (k\lambda_P) & i - \frac{1}{2R} (k\lambda_P)^2 & 0 \\ \frac{1}{2} k\lambda_P & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{P}_k S \\ \hat{T}_k \beta_c \\ \hat{q}_k / \omega \lambda_P \end{bmatrix} = 0 \quad (18)$$

where  $\Delta = \beta_c D_T \lambda_P$ ,  $De = \omega \tau_{relax}$ ,  $R = c/\kappa_T$ . The above relation results in the dispersion relation for a single frequency

$$\left( k^2 \lambda_P^2 - 2i \frac{A+iDe}{B+iDe} \right) \left( \frac{\kappa_T}{c} k^2 \lambda_P^2 - 2i \right) + 2i (\beta_c D_T \lambda_P) k \lambda_P = 0 \quad (19)$$

We can verify that when there is no relaxation  $\tau_{relax} = \infty$  and no thermal diffusion  $\kappa_T = 0$ , the dispersion relation (19) recovers the end-member case (11).

With a finite thermal diffusivity, there are four wavenumbers (named  $k_1, k_2, k_3, k_4$ ). Among the four wavenumbers, two wavenumbers ( $k_2, k_3$ ) that are deviations from the poroelastic endmembers, hence are the main contributing waves driving pressure propagation; two other wavenumbers ( $k_1, k_4$ ) deviate from the thermal diffusion endmember case. Consider an unbound mush column with implicit boundary condition that any perturbation originating from  $z = 0$  eventually vanish at  $z = \pm\infty$ , we can further find that the wavenumbers with positive real parts ( $k_1, k_2$ ) reside in the lower domain  $z < 0$  determining the top-down propagations; wavenumbers ( $k_3, k_4$ ) have negative real values, hence can only reside in the  $z > 0$  domain, determining bottom-up propagations. For the endmember with no thermal-mechanical coupling  $\Delta = 0$ , wavenumbers  $k_1$  and  $k_4$  are symmetric, each having a real part corresponding to a decay length equivalent to the thermal diffusion skin depth  $\lambda_T = \sqrt{2\kappa_T/\omega} = \lambda_P/\sqrt{R}$ ; the wavenumbers  $k_2$  and  $k_3$  also describe symmetric propagation away from  $z = 0$ , each defining a decay length equivalent to the poroelastic diffusion skin depth  $\lambda_P = \sqrt{2c/\omega}$ . With  $\Delta \neq 0$ , the wavenumbers deviate from their endmembers with decay length  $1/Re(k_i)$  ( $i = 1, 2, 3, 4$ ) deviating from the thermal-and-poroelastic diffusion skin depths, which are shown in the main text.

### 1.2.2 Dispersion relation for broadband input

As shown in previous sections, the skin depth  $\lambda_P(\omega)$  and frequency  $\omega$  were chosen as length and time scales for a mush column subjected to harmonic perturbations. When the perturbations are broadband

containing multiple frequencies (hence multiple values for  $\lambda_P$ ), new characteristic scales are required to ensure consistency across all frequency components. For the case of broadband perturbations, the diffusive length and time timescales are determined by the poroelastic diffusivity, the pressure/stress scale is determined by the poroelastic storage coefficient  $S$ , and a temperature scale is determined by the effective thermal expansion coefficient  $\beta_c$ :

$$[l]^2 = c[t], \quad [P] = 1/S, \quad [T] = 1/\beta_c$$

These scales are not frequency dependent, hence persist for different frequency inputs. For a system with viscoelastic relaxation, the (Maxwell) relaxation time  $\tau_{relax}$  is chosen as the timescale, which defines the length scale as  $[l] = \sqrt{c\tau_{relax}}$ . Nondimensionalization of (17) leads to

$$\begin{bmatrix} i\omega - k^2 \frac{BD+i\omega}{AD+i\omega} & -\frac{BD+i\omega}{AD+i\omega} i\omega & 0 \\ k & 0 & 1 \\ 0 & i\omega - \frac{1}{R}k^2 & -\Delta \end{bmatrix} \begin{bmatrix} \hat{P}_k \\ \hat{T}_k \\ \hat{q}_k \end{bmatrix} = 0 \quad (20)$$

where I introduce an additional factor  $D$ , which has the binary value of 0 or 1, indicating if the system has viscoelastic relaxation (i.e., if  $D = 0$  there is no viscoelastic relaxation and length scale is arbitrarily chosen). While  $\Delta$  was defined using the skin depth in the harmonic perturbation case, here it is defined using the new length scale

$$\Delta = \frac{D_T}{[T]/[l]} = D_T \beta_c [l] \quad (21)$$

The dispersion relation resulting from (20) becomes

$$\left(k^2 - \frac{AD + i\omega}{BD + i\omega} \omega i\right) \left(k^2 - R\omega i\right) + \Delta R i \omega k = 0 \quad (22)$$

For gas-bearing magma, the temperature scale  $(1/\beta_c) \gg 1000^\circ C$  (with maximum thermal expansion coefficient and minimum temperature scale corresponding to ideal gas). Realistically, the relevant length-scale for a crustal mush column is no more than 10 km, which I assume to set the upper boundary of the lengthscale  $[l]$ . The thermal gradient constructed by  $[T]$  and  $[l]$  therefore has lower bound  $[T]/[l] \gg 100^\circ C/km$ . Considering a crustal mush column with a background temperature gradient  $D_T$  to be no more than several hundreds  $^\circ C/km$ ,  $\mathcal{O}(\Delta) < 1$  and a maximum value of 2 for  $\Delta$  is assumed in the main text. The dispersion relation, similar to the harmonic perturbation case, predicts four wavenumbers for each frequency  $\omega$ , with  $k_{1,2}$  in the upper domain and  $k_{3,4}$  in lower domain.

### 1.2.3 Derivation for solution in frequency space

Under Fourier transform, an arbitrary function in time and space is expressed as

$$f(t, z) = \int_{-\infty}^{\infty} \hat{f}_\omega(z) e^{i\omega t} d\omega \quad (23)$$

where  $\hat{f}_\omega(z)$  is the Fourier transform of  $f(z, t)$  in the frequency domain. In this section I show the derivation for the Fourier transforms for pressure  $\hat{P}_\omega(z)$ , velocity  $\hat{q}_\omega(z)$  and temperature  $\hat{T}_\omega(z)$  at any target location  $z$  under specific boundary conditions and the dispersion relation obtained in the previous section. The boundary conditions consist of complex amplitudes for pressure and temperature  $\hat{P}_\omega^o, \hat{T}_\omega^o$  at the source  $z = 0$ , which are the Fourier transforms of the time sequences of pressure input  $P_o(t)$  and temperature input  $T_o(t)$ .

As shown in the previous section, there are two wave numbers in each domain, with undetermined amplitudes. The dimensionless solutions for the Fourier transforms therefore can be expressed as

$$\hat{T}_\omega = \hat{T}_{\omega,1}e^{k_1z} + \hat{T}_{\omega,2}e^{k_2z}, \quad \hat{P}_\omega = \gamma \frac{D+i\omega}{AD+i\omega} \hat{P}_\omega^o + \hat{P}_{\omega,1}e^{k_1z} + \hat{P}_{\omega,2}e^{k_2z} \quad (\text{for } z \geq 0) \quad (24a)$$

$$\hat{T}_\omega = \hat{T}_{\omega,1}e^{k_3z} + \hat{T}_{\omega,2}e^{k_4z}, \quad \hat{P}_\omega = \gamma \frac{D+i\omega}{AD+i\omega} \hat{P}_\omega^o + \hat{P}_{\omega,1}e^{k_3z} + \hat{P}_{\omega,2}e^{k_4z} \quad (\text{for } z \leq 0) \quad (24b)$$

where  $k_i$  ( $i = 1, 2, 3, 4$ ) are obtained for  $\omega$  according to (22). Below I show the method for calculating  $\hat{P}_{\omega,i}$  and  $\hat{T}_{\omega,i}$  ( $i=1,2,3,4$ ).

If  $\Delta = 0$  the wavenumbers are the ones for poroelastic diffusion  $k_{2,3}^2 = \frac{AD+i\omega}{BD+i\omega}i\omega$  and thermal diffusion  $k_{1,4}^2 = R\omega i$ . The boundary conditions lead to

$$\hat{T}_{\omega,2} = \hat{T}_{\omega,3} = 0, \quad \hat{T}_{\omega,1} = \hat{T}_{\omega,4} = \hat{T}_\omega^o \quad (25a)$$

$$\hat{P}_{\omega,1} = \frac{\frac{BD+i\omega}{AD+i\omega}}{1 - R\frac{BD+i\omega}{AD+i\omega}} \hat{T}_{\omega,1}, \quad \hat{P}_{\omega,2} = \hat{P}_\omega^o \left(1 - \gamma \frac{D+i\omega}{AD+i\omega}\right) - \hat{P}_{\omega,1} \quad (25b)$$

$$\hat{P}_{\omega,4} = \frac{\frac{BD+i\omega}{AD+i\omega}}{1 - R\frac{BD+i\omega}{AD+i\omega}} \hat{T}_{\omega,4}, \quad \hat{P}_3 = \hat{P}_\omega^o \left(1 - \gamma \frac{D+i\omega}{AD+i\omega}\right) - \hat{P}_{\omega,4} \quad (25c)$$

When  $\Delta \neq 0$ , the boundary conditions can be expressed as

$$\begin{bmatrix} \frac{i\omega - k_1^2/R}{-\Delta k_1} & \frac{i\omega - k_2^2/R}{-\Delta k_2} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{i\omega - k_3^2/R}{-\Delta k_3} & \frac{i\omega - k_4^2/R}{-\Delta k_4} \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{T}_{\omega,1} \\ \hat{T}_{\omega,2} \\ \hat{T}_{\omega,3} \\ \hat{T}_{\omega,4} \end{bmatrix} = \begin{bmatrix} \hat{P}_\omega^o \left(1 - \gamma \frac{D+i\omega}{AD+i\omega}\right) \\ \hat{T}_\omega^o \\ \hat{P}_\omega^o \left(1 - \gamma \frac{D+i\omega}{AD+i\omega}\right) \\ \hat{T}_\omega^o \end{bmatrix} \quad (26)$$

Solving the above equation yields amplitudes  $\hat{T}_{\omega,i}$  ( $i=1,2,3,4$ ). The amplitudes  $\hat{P}_{\omega,i} = \frac{i\omega - k_i^2/R}{-\Delta k_i} \hat{T}_{\omega,i}$  and the amplitudes for Darcy's velocity  $\hat{q}_i = -k_i \hat{P}_{\omega,i} = \frac{i\omega - k_i^2/R}{\Delta} \hat{T}_{\omega,i}$ . The complex amplitudes are summed up according to (24) and result in the final frequency domain solutions  $\hat{P}_\omega, \hat{T}_\omega, \hat{q}_\omega$ .

The above solution scheme applies for broadband source perturbations in pressure and temperature, and one example is shown where the perturbation is only in pressure that evolves as a step function. The time-domain solutions are solved numerically with Fast Fourier Transform, and the input time sequence in pressure is approximated by a square wave for numerical convenience. The numerical Fourier transform and inverse Fourier transforms by Aguilera [2022] are implemented. It is worth noting that to compute the output time sequences, the frequency content for  $\omega \neq 0$  is discarded in the numerical scheme but is added by imposing a zero pressure/velocity initial condition for  $z > 0$ .

## References

- C. A. V. Aguilera, 2022. URL <https://www.mathworks.com/matlabcentral/fileexchange/13327-fouriertransform-m>.
- M. A. Biot. General theory of three-dimensional consolidation. *Journal of applied physics*, 12, 1941.
- A. H.-D. Cheng. *Poroelasticity*. Springer International Publishing, 2016. doi: 10.1007/978-3-319-25202-5.

Y. Liao. The roles of heat and gas in a mushy magma chamber. *Journal of Geophysical Research: Solid Earth*, 127(7):e2022JB024357, 2022. doi: <https://doi.org/10.1029/2022JB024357>. URL <https://agupubs.onlinelibrary.wiley.com/doi/abs/10.1029/2022JB024357>. e2022JB024357 2022JB024357.

D. L. Turcotte and G. Schubert. *Geodynamics*. Cambridge university press, 2002.